Partial differential equations

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Contents

Chapter 1

Preliminaries

We first discuss special functions that arise in the solutions of problems on cylindrical and spherical domains such as Bessel functions, spherical Bessel functions etc. Although these functions can be defined for complex-valued orders and arguments, we shall restrict ourselves to non-negative integer orders and real arguments.

1.1 Bessel and modified Bessel functions

Let λ be a real constant. The Bessel functions of the first and second kind, denoted by $J_k(\lambda r)$ and $Y_k(\lambda r)$ are two linearly independent solutions of the differential equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right)+\left(\lambda^2-\frac{k^2}{r^2}\right)u=0, \tag{1.1}$$

where k is a nonnegative integer (As mentioned above, one can define Bessel functions for even complex-valued k, but we shall restrict ourselves to nonnegative integers) . Thus, the general solution of the above differential equation can be written as $u=c_1J_k(\lambda r)+c_2Y_k(\lambda r)$. The Bessel function of the first kind for a nonnegative integer k is defined in terms of an infinite series as

$$J_k(r):=\left(\frac{r}{2}\right)^k\sum_{l=0}^{\infty}\frac{(-1)^l}{l!(l+k)!}\left(\frac{r^2}{4}\right)^l.$$

The Bessel function of the second kind $Y_k(r)$ can also be expressed as a series, but the expression is more complicated. The Bessel functions $Y_k(r)$ are singular at $r=0$, so that c_2 is set to zero in solutions for domains that include the origin. However, on domains such as the annular space between two circles, both functions need to be included in the general solution. We also have $J_0(0)=1$ and $J_k(0)=0$ for $k\geq 1$, and $\lim_{r\to\infty}J_k(r)=\lim_{r\to\infty}Y_k(r)=0$, $k\geq 0$. The plots of the first few Bessel functions $J_k(r)$ and $Y_k(r)$ are shown in Figs. 1.1.

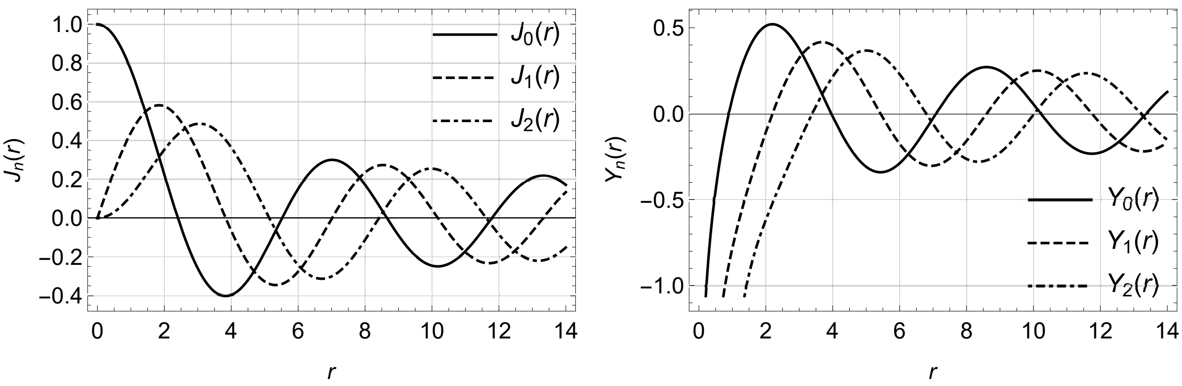

Fig. 1.1: Bessel functions of the first and second kind.

With $H_k(r)$ denoting either $J_k(r)$ or $Y_k(r)$, the Bessel functions satisfy the following recurrence relation:

$$H_{k-1}(r) + H_{k+1}(r) = \frac{2kH_k(r)}{r}. \tag{1.2}$$

Derivative and integral relations involving Bessel functions are

$$\frac{d}{dr}\left[r^k H_k(\beta r)\right] = \beta r^k H_{k-1}(\beta r), \tag{1.3a}$$

$$\frac{d}{dr}\left[r^{-k} H_k(\beta r)\right] = -\beta r^{-k} H_{k+1}(\beta r), \tag{1.3b}$$

$$\frac{d}{dr}\left[r^{k-1} H_k(\beta r)\right] = r^{k-2}\left[\beta r H_{k-1}(\beta r) - H_k(\beta r)\right], \tag{1.3c}$$

$$\int r^k H_{k-1}(\beta r)\, dr = \frac{1}{\beta} r^k H_k(\beta r), \tag{1.3d}$$

$$\int r^{-k} H_{k+1}(\beta r)\, dr = -\frac{1}{\beta} r^{-k} H_k(\beta r). \tag{1.3e}$$

In particular, we have

$$\frac{d}{dr}\left[H_0(\beta r)\right] = -\beta H_1(\beta r), \tag{1.4a}$$

$$\frac{d}{dr}\left[r H_1(\beta r)\right] = \beta r H_0(\beta r), \tag{1.4b}$$

$$\int H_1(\beta r)\, dr = -\frac{1}{\beta} H_0(\beta r), \tag{1.4c}$$

$$\int r H_0(\beta r)\, dr = \frac{1}{\beta} r H_1(\beta r). \tag{1.4d}$$

For arbitrary p, q, and with H_k denoting either $J_k(r)$ or $Y_k(r)$, we have

$$\begin{aligned}\int rH_k(pr)H_k(qr)\,dr &= \frac{r}{p^2-q^2}\left[H_k(pr)\frac{dH_k(qr)}{dr} - H_k(qr)\frac{dH_k(pr)}{dr}\right]\\ &= \frac{r\left[qH_{k-1}(qr)H_k(pr) - pH_{k-1}(pr)H_k(qr)\right]}{p^2-q^2}\\ &= \frac{r\left[pH_{k+1}(pr)H_k(qr) - qH_{k+1}(qr)H_k(pr)\right]}{p^2-q^2}, \quad p \neq q,\\ &= \frac{r^2\left[H_k^2(qr) - H_{k-1}(qr)H_{k+1}(qr)\right]}{2}, \quad p = q.\end{aligned} \tag{1.5}$$

Thus, if λ_m, $m = 1, 2, \ldots, \infty$, denote the roots of $J_k(x) = 0$, and δ_{mn} denotes the Kronecker delta, then from Eqns. (1.2) and (1.5), it follows that the Bessel functions $J_k(\cdot)$ are orthogonal in the following sense:

$$\int_0^R rJ_k\left(\frac{\lambda_m r}{R}\right)J_k\left(\frac{\lambda_n r}{R}\right)dr = \frac{R^2}{2}\left[J_{k+1}(\lambda_m)\right]^2\delta_{mn} = \frac{R^2}{2}\left[J_{k-1}(\lambda_m)\right]^2\delta_{mn}. \tag{1.6}$$

If λ_m, $m = 1, 2, \ldots, \infty$, denote the roots of $J_k'(x) = 0$, then from Eqns. (1.1) and (1.5), we get the orthogonality relation (with no sum on n)

$$\int_0^R \frac{1}{r}\left[\left(\frac{\lambda_m r}{R}\right)^2 - k^2\right]J_k\left(\frac{\lambda_m r}{R}\right)dr = 0, \tag{1.7a}$$

$$\int_0^R rJ_k\left(\frac{\lambda_m r}{R}\right)J_k\left(\frac{\lambda_n r}{R}\right)dr = \frac{R^2(\lambda_n^2 - k^2)J_k^2(\lambda_n)\delta_{mn}}{2\lambda_n^2}. \tag{1.7b}$$

The modified Bessel functions of the first and second kind $I_n(\lambda r)$ and $K_n(\lambda r)$ are the solutions of the equation

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right) - \left(\lambda^2 + \frac{n^2}{r^2}\right)u = 0. \tag{1.8}$$

Similar to $Y_n(r)$, the functions $K_n(r)$ are unbounded at the origin. We also have $I_0(0) = 1$, $I_n(0) = 0$ for $n \geq 1$, $\lim_{r\to\infty} I_n(r) = \infty$, and $\lim_{r\to\infty} K_n(r) = 0$, $n \geq 0$. The plots of the first few modified Bessel functions are shown in Fig. 1.2.

The modified Bessel functions satisfy the following recurrence relations:

$$I_{n-1}(r) - I_{n+1}(r) = \frac{2nI_n(r)}{r}, \tag{1.9a}$$

$$K_{n+1}(r) - K_{n-1}(r) = \frac{2nK_n(r)}{r}. \tag{1.9b}$$

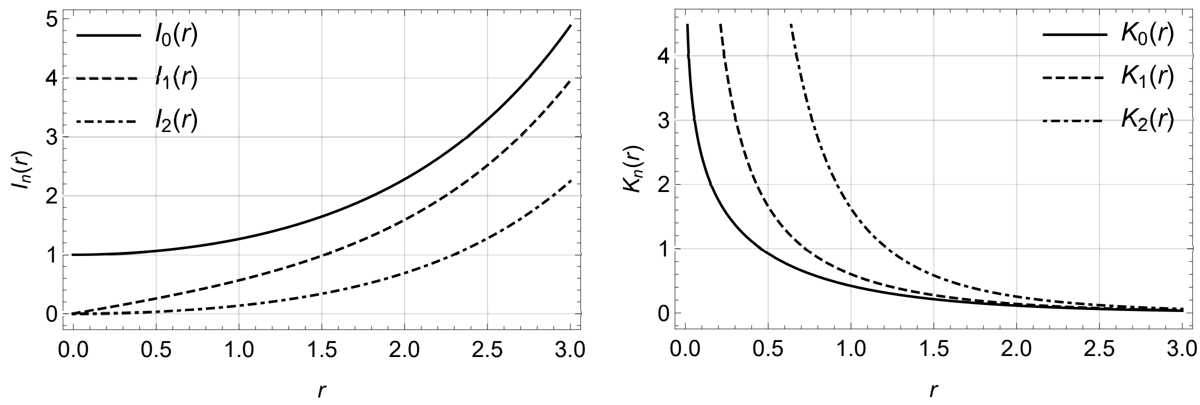

Fig. 1.2: Modified Bessel functions of the first and second kind.

Derivative relations involving modified Bessel functions are

$$\frac{d}{dr}\left[r^n I_n(\beta r)\right] = \beta r^n I_{n-1}(\beta r), \tag{1.10a}$$

$$\frac{d}{dr}\left[r^{-n} I_n(\beta r)\right] = \beta r^{-n} I_{n+1}(\beta r), \tag{1.10b}$$

$$\frac{d}{dr}\left[r^n K_n(\beta r)\right] = -\beta r^n K_{n-1}(\beta r), \tag{1.10c}$$

$$\frac{d}{dr}\left[r^{-n} K_n(\beta r)\right] = -\beta r^{-n} K_{n+1}(\beta r), \tag{1.10d}$$

$$\frac{d}{dr}\left[r^{n-1} I_n(\beta r)\right] = r^{n-2}\left[\beta r I_{n-1}(\beta r) - I_n(\beta r)\right], \tag{1.10e}$$

$$\frac{d}{dr}\left[r^{n-1} K_n(\beta r)\right] = -r^{n-2}\left[\beta r K_{n-1}(\beta r) + K_n(\beta r)\right]. \tag{1.10f}$$

In particular, we have

$$\frac{d}{dr}\left[I_0(\beta r)\right] = \beta I_1(\beta r), \tag{1.11a}$$

$$\frac{d}{dr}\left[r I_1(\beta r)\right] = \beta r I_0(\beta r), \tag{1.11b}$$

$$\frac{d}{dr}\left[K_0(\beta r)\right] = -\beta K_1(\beta r), \tag{1.11c}$$

$$\frac{d}{dr}\left[r K_1(\beta r)\right] = -\beta r K_0(\beta r), \tag{1.11d}$$

$$\int I_1(\beta r)\, dr = \frac{1}{\beta} I_0(\beta r), \tag{1.11e}$$

$$\int rI_0(\beta r)\,dr = \frac{rI_1(\beta r)}{\beta}, \tag{1.11f}$$

$$\int K_1(\beta r)\,dr = -\frac{1}{\beta}K_0(\beta r), \tag{1.11g}$$

$$\int rK_0(\beta r)\,dr = -\frac{rK_1(\beta r)}{\beta}. \tag{1.11h}$$

1.2 Spherical Bessel and modified spherical Bessel functions

The spherical Bessel functions of the first and second kind, denoted by $j_k(\lambda r)$ and $y_k(\lambda r)$ are two linearly independent solutions of the differential equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right) + \left(\lambda^2 - \frac{k(k+1)}{r^2}\right)u = 0. \tag{1.12}$$

Thus, the general solution of the above differential equation can be written as $u = c_1 j_n(\lambda r) + c_2 y_n(\lambda r)$. Similar to the Bessel functions $Y_k(r)$, the spherical Bessel functions $y_k(r)$ are singular at $r = 0$, so that c_2 is set to zero in solutions for domains that include the origin. However, on domains such as the annular space between two spheres, both functions need to be included in the general solution. We also have $j_0(0) = 1$ and $j_k(0) = 0$ for $k \geq 1$, and $\lim_{r\to\infty} j_n(r) = \lim_{r\to\infty} y_n(r) = 0$, $n \geq 0$.

The first few spherical Bessel functions are

$$j_0(kr) = \frac{\sin(kr)}{kr}, \qquad y_0(kr) = -\frac{\cos(kr)}{kr},$$

$$j_1(kr) = \frac{\sin(kr)}{(kr)^2} - \frac{\cos(kr)}{kr}, \qquad y_1(kr) = -\frac{\cos(kr)}{(kr)^2} - \frac{\sin(kr)}{kr},$$

$$j_2(kr) = \left[\frac{3}{(kr)^3} - \frac{1}{kr}\right]\sin(kr) - \frac{3\cos(kr)}{(kr)^2}, \quad y_2(kr) = \left[\frac{1}{kr} - \frac{3}{(kr)^3}\right]\cos(kr) - \frac{3\sin(kr)}{(kr)^2},$$

and their plots are shown in Fig. 1.3.

The spherical Bessel functions are related to the Bessel functions by the relations

$$j_n(r) := \sqrt{\frac{\pi}{2r}}J_{n+\frac{1}{2}}(r) = (-1)^n\sqrt{\frac{\pi}{2r}}Y_{-n-\frac{1}{2}}(r),$$

$$y_n(r) := \sqrt{\frac{\pi}{2r}}Y_{n+\frac{1}{2}}(r) = (-1)^{n+1}\sqrt{\frac{\pi}{2r}}J_{-n-\frac{1}{2}}(r).$$

With $h \equiv j, y$, the spherical Bessel functions satisfy the following recurrence relation:

$$h_{n-1}(r) + h_{n+1}(r) = \frac{2n+1}{r}h_n(r). \tag{1.13}$$

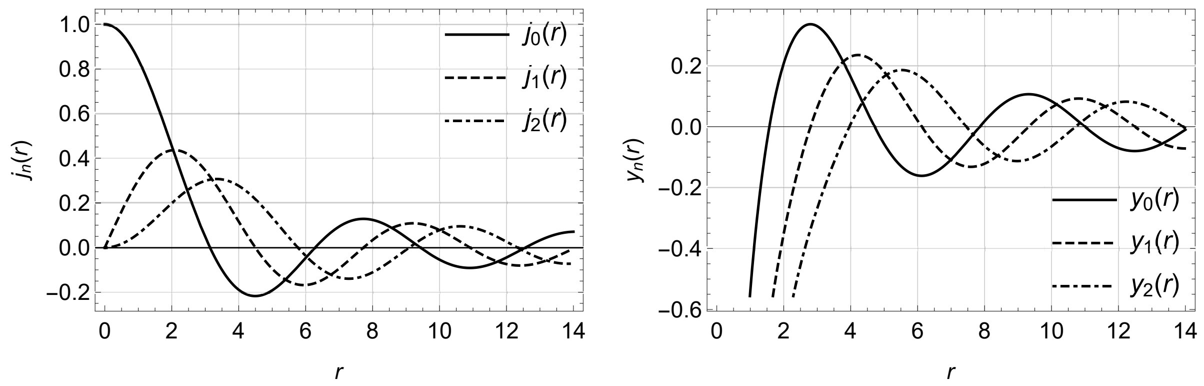

Fig. 1.3: Spherical Bessel functions of the first and second kind.

Derivative and integral relations involving spherical Bessel functions are

$$\frac{d}{dr}\left[r^{k+1}h_k(\beta r)\right] = \beta r^{k+1}h_{k-1}(\beta r), \tag{1.14a}$$

$$\frac{d}{dr}\left[r^{-k}h_k(\beta r)\right] = -\beta r^{-k}h_{k+1}(\beta r), \tag{1.14b}$$

$$\frac{d}{dr}\left[r^{n-1}h_n(\beta r)\right] = r^{n-2}\left[\beta r h_{n-1}(\beta r) - 2h_n(\beta r)\right], \tag{1.14c}$$

$$\int r^{n+1}h_{n-1}(\beta r)\,dr = \frac{1}{\beta}r^{n+1}h_n(\beta r), \tag{1.14d}$$

$$\int r^{-n}h_{n+1}(\beta r)\,dr = -\frac{1}{\beta}r^{-n}h_n(\beta r). \tag{1.14e}$$

In particular, we have

$$\frac{d}{dr}\left[h_0(\beta r)\right] = -\beta h_1(\beta r), \tag{1.15a}$$

$$\frac{d}{dr}\left[r^2 h_1(\beta r)\right] = \beta r^2 h_0(\beta r), \tag{1.15b}$$

$$\int h_1(\beta r)\,dr = -\frac{1}{\beta}h_0(\beta r), \tag{1.15c}$$

$$\int r^2 h_0(\beta r)\,dr = \frac{1}{\beta}r^2 h_1(\beta r). \tag{1.15d}$$

For arbitrary p, q, and with h_k denoting either $j_k(r)$ or $y_k(r)$, we have

$$\begin{aligned}\int r^2 h_k(pr)h_k(qr)\,dr &= \frac{r^2}{p^2-q^2}\left[h_k(pr)\frac{dh_k(qr)}{dr} - h_k(qr)\frac{dh_k(pr)}{dr}\right]\\ &= \frac{r^2\left[qh_{k-1}(qr)h_k(pr) - ph_{k-1}(pr)h_k(qr)\right]}{p^2-q^2}\\ &= \frac{r^2\left[ph_{k+1}(pr)h_k(qr) - qh_{k+1}(qr)h_k(pr)\right]}{p^2-q^2}\\ &= \frac{r^3\left[h_k^2(qr) - h_{k-1}(qr)h_{k+1}(qr)\right]}{2}, \quad p=q.\end{aligned} \tag{1.16}$$

If λ_m, $m = 1, 2, \ldots, \infty$, denote the roots of $j_k(x) = 0$, the orthogonality property that follows from Eqns. (1.13) and (1.16) is given by

$$\int_0^R r^2 j_k\left(\frac{\lambda_m r}{R}\right) j_k\left(\frac{\lambda_n r}{R}\right) dr = \frac{R^3\left[j_{k+1}(\lambda_m)\right]^2\delta_{mn}}{2} = \frac{R^3\left[j_{k-1}(\lambda_m)\right]^2\delta_{mn}}{2}, \quad \text{(no sum on } n). \tag{1.17}$$

If λ_n, $n = 1, 2, \ldots, \infty$, denote the roots of $j_k'(x) = 0$, then from Eqns. (1.12) and (1.16), we get the orthogonality relation (with no sum on n)

$$\int_0^R \left[\left(\frac{\lambda_n r}{R}\right)^2 - k(k+1)\right] j_k\left(\frac{\lambda_n r}{R}\right) dr = 0, \tag{1.18a}$$

$$\int_0^R r^2 j_k\left(\frac{\lambda_m r}{R}\right) j_k\left(\frac{\lambda_n r}{R}\right) dr = \frac{R^3\delta_{mn}[\lambda_n^2 - k(k+1)]j_k^2(\lambda_n)}{2\lambda_n^2}. \tag{1.18b}$$

The modified spherical Bessel functions of the first and second kind $i_n(\lambda r)$ and $k_n(\lambda r)$ are two linearly independent solutions of the differential equation

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{du}{dr}\right) - \left(\lambda^2 + \frac{n(n+1)}{r^2}\right)u = 0, \tag{1.19}$$

where n is a nonnegative integer. Similar to $K_n(r)$, the functions $k_n(r)$ are unbounded at the origin. We also have $i_0(0) = 1$, $i_n(0) = 0$ for $n \geq 1$, $\lim_{r\to\infty} i_n(r) = \infty$, and $\lim_{r\to\infty} k_n(r) = 0$, $n \geq 0$. The plots of the first few modified spherical Bessel functions are shown in Fig. 1.4.

The modified spherical Bessel functions satisfy the following recurrence relations:

$$i_{n-1}(r) - i_{n+1}(r) = \frac{2n+1}{r} i_n(r), \tag{1.20a}$$

$$k_{n+1}(r) - k_{n-1}(r) = \frac{2n+1}{r} k_n(r). \tag{1.20b}$$

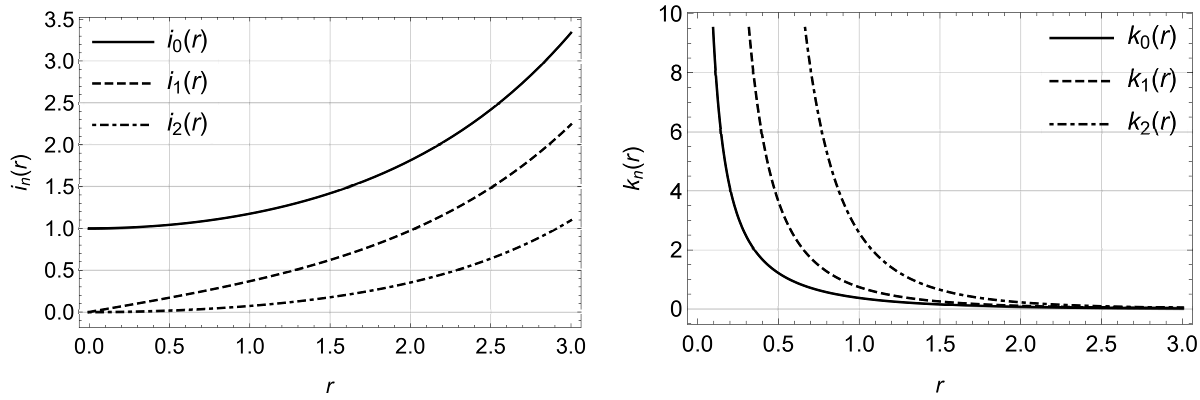

Fig. 1.4: Modified spherical Bessel functions of the first and second kind.

Derivative relations involving modified spherical Bessel functions are

$$\frac{d}{dr}\left[r^{n+1}i_n(\beta r)\right] = \beta r^{n+1}i_{n-1}(\beta r), \tag{1.21a}$$

$$\frac{d}{dr}\left[r^{-n}i_n(\beta r)\right] = \beta r^{-n}i_{n+1}(\beta r), \tag{1.21b}$$

$$\frac{d}{dr}\left[r^{n+1}k_n(\beta r)\right] = -\beta r^{n+1}k_{n-1}(\beta r), \tag{1.21c}$$

$$\frac{d}{dr}\left[r^{-n}k_n(\beta r)\right] = -\beta r^{-n}k_{n+1}(\beta r). \tag{1.21d}$$

In particular, we have

$$\frac{d}{dr}\left[i_0(\beta r)\right] = \beta i_1(\beta r), \tag{1.22a}$$

$$\frac{d}{dr}\left[r^2 i_1(\beta r)\right] = \beta r^2 i_0(\beta r), \tag{1.22b}$$

$$\frac{d}{dr}\left[k_0(\beta r)\right] = -\beta k_1(\beta r), \tag{1.22c}$$

$$\frac{d}{dr}\left[r^2 k_1(\beta r)\right] = -\beta r^2 k_0(\beta r), \tag{1.22d}$$

$$\int i_1(\beta r)\, dr = \frac{1}{\beta} i_0(\beta r), \tag{1.22e}$$

$$\int r^2 i_0(\beta r)\, dr = \frac{1}{\beta} r^2 i_1(\beta r), \tag{1.22f}$$

$$\int k_1(\beta r)\, dr = -\frac{1}{\beta} k_0(\beta r), \tag{1.22g}$$

$$\int r^2 k_0(\beta r)\, dr = -\frac{1}{\beta} r^2 k_1(\beta r). \tag{1.22h}$$

1.3 Legendre functions

The Legendre equation is given by

$$\frac{d}{d\xi}\left[(1-\xi^2)\frac{dH}{d\xi}\right] + \left[k(k+1) - \frac{m^2}{1-\xi^2}\right] H = 0, \tag{1.23}$$

The solution is given by

$$H(\xi) = c_1 P_k^{(m)}(\xi) + c_2 Q_k^{(m)}(\xi),$$

where $P_k^{(m)}$ and $Q_k^{(m)}$ are the associated Legendre functions of the first and second kind, respectively. The solutions $P_k^{(m)}(\xi)$ diverge for $\xi = -1$ unless k is a non-negative integer. Since, in this work, we treat only spherical domains (and not, for example, conical domains, where the axis $\xi = -1$ may not be a part of the domain), we henceforth assume the 'k' in $P_k^{(m)}(\xi)$ to be a non-negative integer. One can also show that m can assume only discrete values in the range $[-n, n]$. For $m = 0$, Eqn. (1.23) reduces to

$$\frac{d}{d\xi}\left[(1-\xi^2)\frac{dH}{d\xi}\right] + k(k+1)H = 0. \tag{1.24}$$

One set of solutions of the above equation, known as the Legendre polynomials, is given by the Rodrigues formula as

$$P_k(\xi) = \frac{1}{2^k k!}\frac{d^k}{d\xi^k}(\xi^2 - 1)^k. \tag{1.25}$$

Thus, again, we see that the solution is obtained in the form of a series in ξ. For domains that include the axis in axisymmetric problems, the functions $Q_k(\xi)$ can be excluded since they are singular.

The first few Legendre polynomials and functions of the second kind are

$$P_0(\xi) = 1, \qquad Q_0(\xi) = \frac{1}{2}\log\left(\frac{1+\xi}{1-\xi}\right),$$

$$P_1(\xi) = \xi, \qquad Q_1(\xi) = \frac{\xi}{2}\log\left(\frac{1+\xi}{1-\xi}\right) - 1,$$

$$P_2(\xi) = \frac{1}{2}(3\xi^2 - 1), \qquad Q_2(\xi) = \frac{3\xi^2 - 1}{4}\log\left(\frac{1+\xi}{1-\xi}\right) - \frac{3\xi}{2},$$

$$P_3(\xi) = \frac{1}{2}(5\xi^3 - 3\xi), \qquad Q_3(\xi) = \frac{5\xi^3 - 3\xi}{4}\log\left(\frac{1+\xi}{1-\xi}\right) + \frac{4 - 15\xi^2}{6},$$

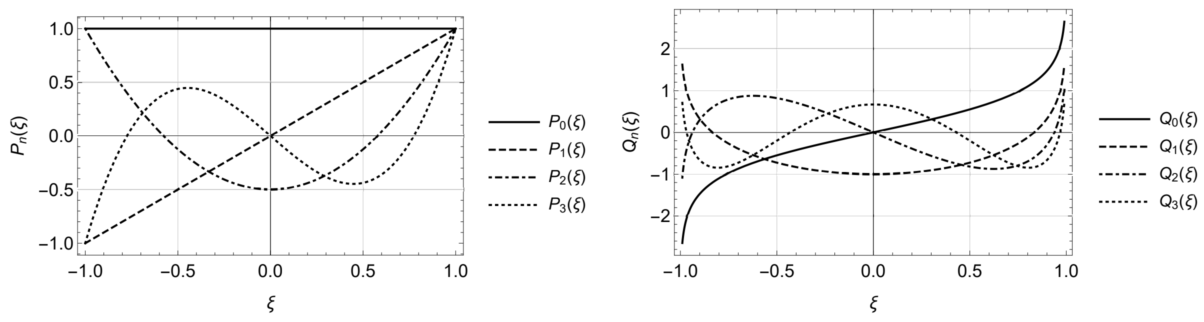

Fig. 1.5: Legendre polynomials and functions of the second kind.

$$P_4(\xi) = \frac{1}{8}(35\xi^4 - 30\xi^2 + 3), \quad Q_4(\xi) = \frac{35\xi^4 - 30\xi^2 + 3}{16} \log\left(\frac{1+\xi}{1-\xi}\right) + \frac{55\xi - 105\xi^3}{24}.$$

Note that $Q_n(\xi)$ is singular at $\xi = \pm 1$ for all values of n. We also have

$$P_n^{(m)}(1) = \begin{cases} 0 & m \geq 1, \\ 1 & m = 0, \end{cases} \quad m, n \in \mathbb{N},$$

$$P_n^{(m)}(-1) = \begin{cases} 0 & m \geq 1, \\ (-1)^n & m = 0, \end{cases} \quad m, n \in \mathbb{N}.$$

The plots of the first few Legendre polynomials and functions of the second kind are shown in Fig. 1.5.

With $H \equiv P, Q$, the associated Legendre functions satisfy the following recurrence relation:

$$(n - m + 1)H_{n+1}^{(m)}(\xi) = (2n + 1)\xi H_n^{(m)}(\xi) - (n + m)H_{n-1}^{(m)}(\xi), \tag{1.26}$$

which for $m = 0$ reduces to

$$(n + 1)H_{n+1}(\xi) = (2n + 1)\xi H_n(\xi) - nH_{n-1}(\xi) \tag{1.27}$$

The associated Legendre polynomials of the first kind satisfy the following orthogonality relation:

$$\int_{-1}^{1} P_k^{(m)}(\xi) P_l^{(m)}(\xi)\, d\xi = \frac{2(l + m)!\delta_{kl}}{(2l + 1)(l - m)!},$$

which for $m = 0$ reduces to (with no sum on l)

$$\int_{-1}^{1} P_k(\xi) P_l(\xi)\, d\xi = \frac{2\delta_{kl}}{2l + 1}. \tag{1.28}$$

We also have

$$\int_{-1}^{1} P_n(\xi)\, d\xi = \begin{cases} 2, & n = 0, \\ 0, & n \geq 1. \end{cases} \tag{1.29}$$

The derivative of the associated Legendre functions is given by

$$(1-\xi^2)\frac{dP_n^{(m)}(\xi)}{d\xi} = (1+n)\xi P_n^{(m)}(\xi) + (m-n-1)P_{n+1}^{(m)}(\xi), \tag{1.30}$$

which when combined with Eqn. (1.27) yields for $m = 0$,

$$(1-\xi^2)\frac{dP_n(\xi)}{d\xi} = n\left[P_{n-1}(\xi) - \xi P_n(\xi)\right] = (n+1)\left[\xi P_n(\xi) - P_{n+1}(\xi)\right]. \tag{1.31}$$

1.4 Fourier and Hankel transforms

For all the relations in this section, n is an arbitrary real number.

Fourier sine transform

If $f(x)$ is a function defined on the domain $[0,\infty)$, and if

$$\int_0^\infty A(\lambda)\sin(\lambda x)\, d\lambda = f(x), \tag{1.32}$$

then

$$A(\lambda) = \frac{2}{\pi}\int_0^\infty f(\xi)\sin\lambda\xi\, d\xi, \tag{1.33}$$

Fourier cosine transform

If $f(x)$ is a function defined on the domain $[0,\infty)$, and if

$$\int_0^\infty A(\lambda)\cos(\lambda x)\, d\lambda = f(x), \tag{1.34}$$

then

$$A(\lambda) = \frac{2}{\pi}\int_0^\infty f(\xi)\cos\lambda\xi\, d\xi. \tag{1.35}$$

Fourier transform

If $f(x)$ is a function defined on the domain $(-\infty,\infty)$, and if

$$\int_0^\infty \left[A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)\right] d\lambda = f(x), \tag{1.36}$$

then

$$A(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\cos\lambda\xi\, d\xi,$$
$$B(\lambda) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(\xi)\sin\lambda\xi\, d\xi. \tag{1.37}$$

Hankel transform

If $f(r)$ is a function defined on $[0,\infty)$, and if

$$\int_0^{\infty} A(\lambda)J_n(\lambda r)\, d\lambda = f(r), \tag{1.38}$$

then

$$A(\lambda) = \lambda\int_0^{\infty} \hat{r}f(\hat{r})J_n(\lambda\hat{r})\, d\hat{r}. \tag{1.39}$$

Spherical Hankel transform

If

$$\int_0^{\infty} A(\lambda)j_n(\lambda r)\, d\lambda = f(r), \tag{1.40}$$

then

$$A(\lambda) = \frac{2\lambda^2}{\pi}\int_0^{\infty} \hat{r}^2 f(\hat{r})j_n(\lambda\hat{r})\, d\hat{r}. \tag{1.41}$$

1.5 Fourier, Fourier–Bessel and Legendre series

This section is partly based on [1]. We shall consider separable solutions to typical elliptic (e.g., steady-state heat conduction equation), parabolic (e.g., transient heat conduction equation) and hyperbolic (e.g., acoustic wave equation) partial differential equations.

1.5.1 Separable solutions to $\boldsymbol{\nabla}^2 T = 0$

Consider the Laplace equation $\boldsymbol{\nabla}^2 T = 0$. The functions $T(x,y)$ that satisfy the Laplace equation are said to be *harmonic*. These functions are generally represented in the form of an infinite series or an integral. Imposing the boundary condition on the infinite series solution on a given surface often results in a Fourier, Fourier–Bessel or Legendre series.

Consider the equation

$$\boldsymbol{\nabla}^2 T \equiv \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0. \tag{1.42}$$

Suppose that this governing equation is to be solved subject to appropriate Dirichlet or Neumann boundary conditions on the edges of a rectangular domain $[-L,L]\times[0,H]$. By the standard separation of variables method, we write $T = X(x)Y(y)$ which on substituting into Eqn. (1.42) yields

$$\frac{X''x}{X(x)} = -\frac{Y''(y)}{Y(y)} = -k^2, \tag{1.43}$$

where k is a constant. By superposing the solutions obtained for the infinite separation constants k, we get the general solution as

$$\begin{aligned} T &= c_0 + c_1 x + c_2 y + c_3 xy \\ &+ \sum_{n=1}^{\infty} \Big\{ A_n \sin(k_{1n}x)\sinh(k_{1n}y) + B_n \sin(k_{2n}x)\cosh(k_{2n}y) \\ &+ C_n \cos(k_{3n}x)\sinh(k_{3n}y) + D_n \cos(k_{4n}x)\cosh(k_{4n}y) \\ &+ E_n \sinh(k_{5n}x)\sin(k_{5n}y) + F_n \sinh(k_{6n}x)\cos(k_{6n}y) \\ &+ G_n \cosh(k_{7n}x)\sin(k_{7n}y) + H_n \cosh(k_{8n}x)\cos(k_{8n}y) \Big\}. \end{aligned} \tag{1.44}$$

The nonseries terms in Eqn. (1.44) correspond to $k = 0$ in Eqn. (1.43), and are *linearly independent* of the series terms which correspond to a nonzero k value. We shall see that these nonseries terms play a key role in what follows.

With $\gamma_{mn}^2 = \alpha_m^2 + \beta_n^2$, a general three-dimensional separable solution to the Laplace equation is

$$\begin{aligned} T &= c_0 + c_1 x + c_2 y + c_3 z + c_4 xy + c_5 yz + c_6 xz + c_7 xyz \\ &+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \cos(\alpha_m x)\cos(\beta_n y)\cosh(\gamma_{mn} z), \end{aligned} \tag{1.45}$$

and similar terms in sin(.), sinh(.) etc. which need to be added to the above solution.

If we want represent an odd function $f(x)$ on $[-L, L]$ using $\sin(n\pi x/L)$, then standard Fourier theory would suggest writing it as

$$\sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} = f(x), \tag{1.46}$$

with

$$C_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx. \tag{1.47}$$

However, from Eqn. (1.44) evaluated at constant y, we see that we should write it as

$$C_0 x + \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L} = f(x), \tag{1.48}$$

with

$$\begin{aligned} C_0 &= \frac{f(L)}{L}, \\ \tilde{f}(x) &= f(x) - C_0 x, \\ C_n &= \frac{1}{L}\int_{-L}^{L} \tilde{f}(x)\sin\frac{n\pi x}{L}\,dx. \end{aligned} \tag{1.49}$$

As an example, if $f(x) = (x/L)^7$, then by the classical representation given by Eqn. (1.46), we have

$$C_n = \frac{2(-1)^{n+1}\left[(n\pi)^6 - 42(n\pi)^4 + 840(n\pi)^2 - 5040\right]}{(n\pi)^7}. \tag{1.50}$$

By the proposed representation given by Eqn. (1.48), we have

$$\begin{aligned} C_0 &= \frac{1}{L}, \\ C_n &= \frac{84(-1)^n\left[(n\pi)^4 - 20(n\pi)^2 + 120\right]}{(n\pi)^7}. \end{aligned} \tag{1.51}$$

Thus, the classical and proposed representations of $(x/L)^7$ are

$$\left(\frac{x}{L}\right)^7 \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}\left[(n\pi)^6 - 42(n\pi)^4 + 840(n\pi)^2 - 5040\right]\sin(n\pi x/L)}{(n\pi)^7}, \tag{1.52a}$$

$$\left(\frac{x}{L}\right)^7 \sim \frac{x}{L} + \sum_{n=1}^{\infty} \frac{84(-1)^n\left[(n\pi)^4 - 20(n\pi)^2 + 120\right]\sin(n\pi x/L)}{(n\pi)^7}. \tag{1.52b}$$

Fig. 1.6 clearly shows the rapid convergence of the proposed representation as compared to the classical one besides, of course, removing the error at the endpoints. Now consider the representation of an even function $f(x)$ over $[-L, L]$ by $\cos[(2n-1)\pi x/(2L)]$. By standard Fourier theory, we have

$$\sum_{n=1}^{\infty} C_n \cos\frac{(2n-1)\pi x}{2L} = f(x),$$

with

$$C_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{(2n-1)\pi x}{2L}\,dx.$$

Instead, we propose

$$C_0 + \sum_{n=1}^{\infty} C_n \cos\frac{(2n-1)\pi x}{2L} = f(x), \tag{1.53}$$

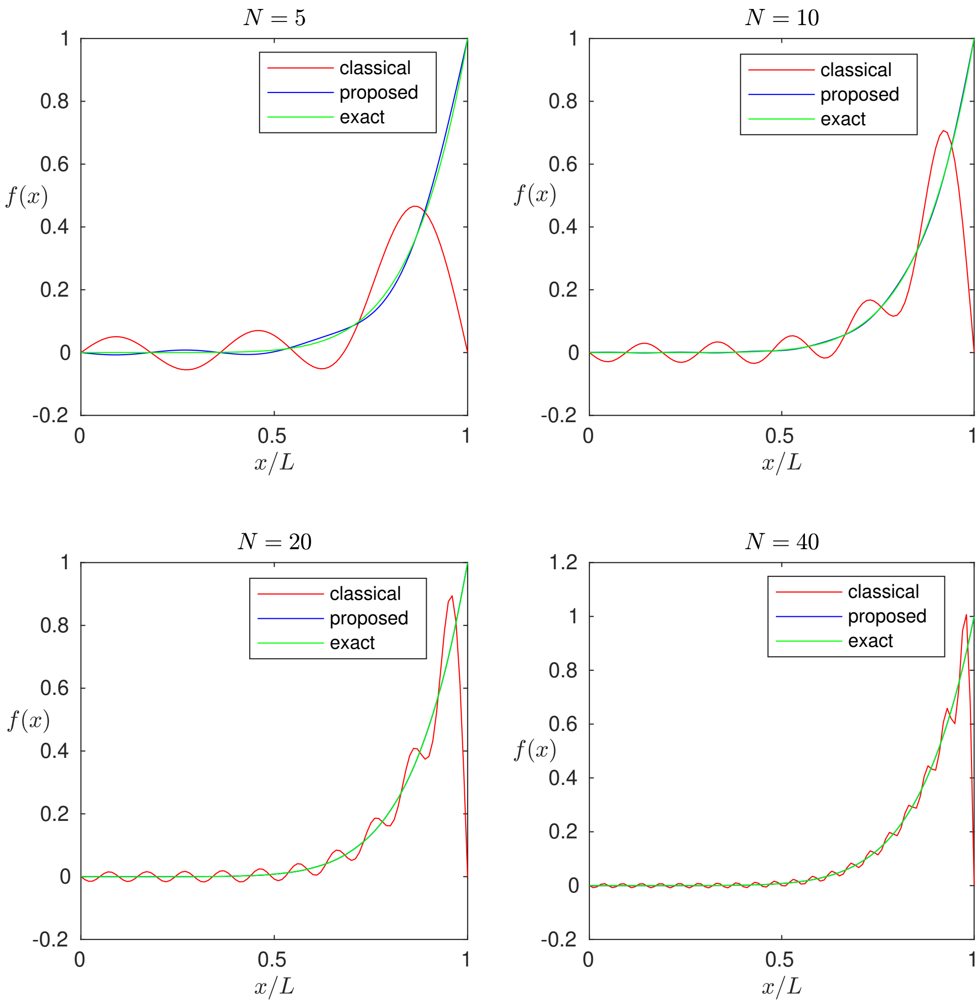

Fig. 1.6: Plots of the representations in Eqns. (1.52) for different number of terms N in the Fourier series.

with

$$C_0 = f(L),$$
$$\tilde{f}(x) := f(x) - f(L),$$
$$C_n = \frac{1}{L}\int_{-L}^{L} \tilde{f}(x)\cos\frac{(2n-1)\pi x}{2L}\,dx.$$

In the classical framework, the C_0 term is present when the even function is being approximated by $\cos(n\pi x/L)$, but not when it is being approximated by $\cos[(2n-1)\pi x/(2L)]$.

As an example, if $f(x) = (x/L)^6$, then by the classical representation, and with $\beta_n := (2n-1)\pi$, we have

$$C_n = \frac{4(-1)^{n+1}\left(\beta_n^6 - 120\beta_n^4 + 5760\beta_n^2 - 46080\right)}{\beta_n^7}.$$

By the proposed one, we have

$$C_0 = 1,$$
$$C_n = \frac{480(-1)^n\left(\beta_n^4 - 48\beta_n^2 + 384\right)}{\beta_n^7}.$$

Thus, the two representations (classical and proposed) are

$$\left(\frac{x}{L}\right)^6 \sim \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}\left(\beta_n^6 - 120\beta_n^4 + 5760\beta_n^2 - 46080\right)\cos[\beta_n x/(2L)]}{\beta_n^7}, \tag{1.54a}$$

$$\left(\frac{x}{L}\right)^6 \sim 1 + \sum_{n=1}^{\infty} \frac{480(-1)^n\left(\beta_n^4 - 48\beta_n^2 + 384\right)\cos[\beta_n x/(2L)]}{\beta_n^7}. \tag{1.54b}$$

Once again, Fig. 1.7 clearly shows the rapid convergence of the proposed representation as compared to the classical one besides, of course, removing the error at the endpoints.

Now consider the representation of an odd function $f(x)$ defined on $[-L, L]$ by $\sin[(2n-1)\pi x/(2L)]$. Since we use the derivative of $f(x)$ evaluated at $x = L$, we need to exercise care here. For example, the function

$$f(x) = x\sqrt{1 - \left(\frac{x}{L}\right)^2}, \tag{1.55}$$

is bounded on $[-L, L]$, but its derivative is not bounded at $x = L$. We do not consider such functions, and restrict ourselves to continuously differentiable functions, i.e., $f(x) \in C^1$. Thus, we now propose

$$C_1 x + \sum_{n=1}^{\infty} C_n \sin\frac{(2n-1)\pi x}{2L} = f(x), \tag{1.56}$$

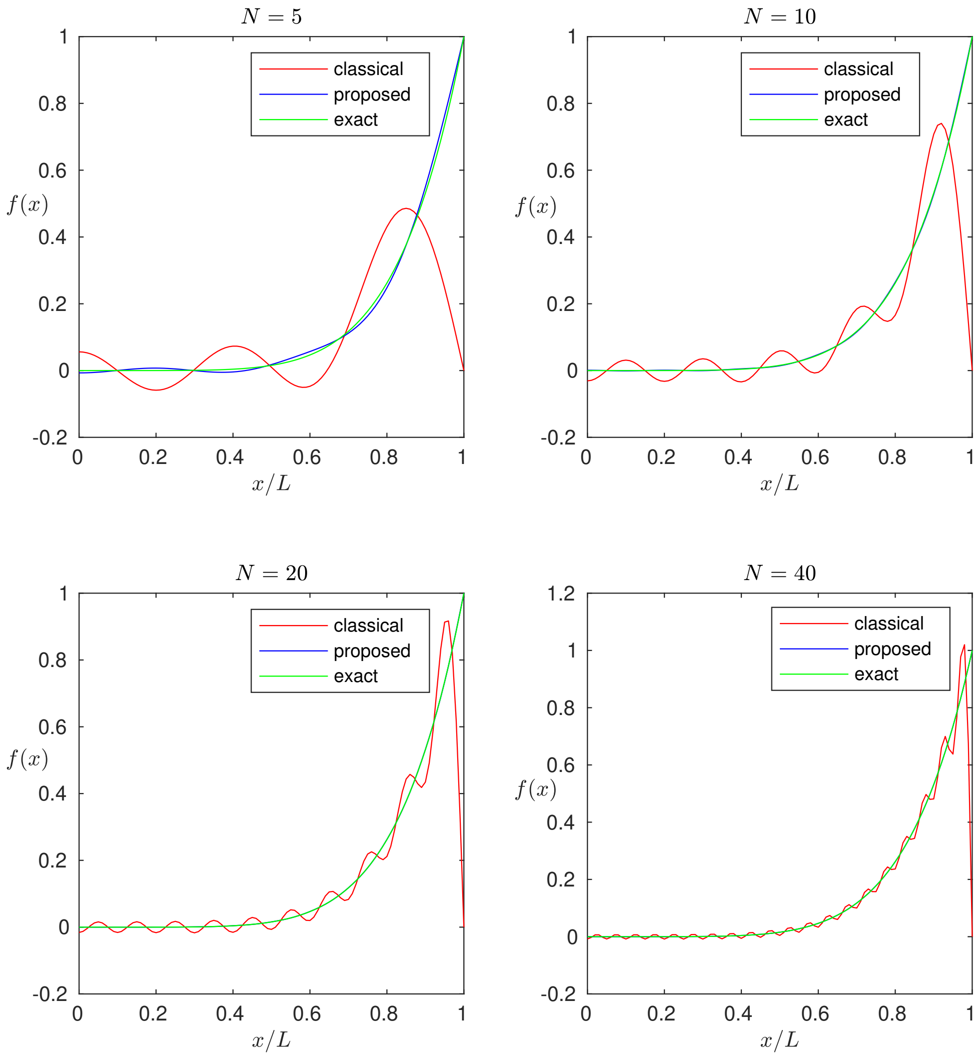

Fig. 1.7: Plots of the representations in Eqns. (1.54) for different number of terms N in the Fourier series.

with

$$\begin{aligned} C_1 &= f'(L), \\ \tilde{f}(x) &= f(x) - C_1 x, \\ C_n &= \frac{1}{L}\int_{-L}^{L} \tilde{f}(x)\sin\frac{(2n-1)\pi x}{2L}\,dx. \end{aligned} \tag{1.57}$$

As an example, the classical and proposed approximations for $(x/L)^7$ with $\beta_n := (2n-1)\pi$ are

$$\left(\frac{x}{L}\right)^7 \sim \sum_{n=1}^{\infty} \frac{56(-1)^{n+1}\left(\beta_n^6 - 120\beta_n^4 + 5760\beta_n^2 - 46080\right)\sin[\beta_n x/(2L)]}{\beta_n^8}, \tag{1.58a}$$

$$\left(\frac{x}{L}\right)^7 \sim \frac{7x}{L} + \sum_{n=1}^{\infty} \frac{6720(-1)^{n}\left(\beta_n^4 - 48\beta_n^2 + 384\right)\sin[\beta_n x/(2L)]}{\beta_n^8}. \tag{1.58b}$$

While representing an even function on $[-L, L]$ with $\cos(n\pi x/L)$, the conventional representation given by

$$C_0 + \sum_{n=1}^{\infty} C_n \cos\frac{n\pi x}{L} = f(x), \tag{1.59}$$

with

$$\begin{aligned} C_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)\,dx, \\ C_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx, \end{aligned}$$

already has the constant term C_0, and so in this case, no additional terms need to be added.

As another example, if one wants to approximate the function $f(x)$ on $[0, L]$ using $\sin(n\pi x/L)$, then we get

$$f(x) \sim f(0) + \frac{[f(L) - f(0)]x}{L} + \sum_{n=1}^{\infty} C_n \sin\frac{n\pi x}{L}, \tag{1.60}$$

where

$$\begin{aligned} \tilde{f}(x) &:= f(x) - f(0) - \frac{[f(L) - f(0)]x}{L}, \\ C_n &= \frac{2}{L}\int_{0}^{L} \tilde{f}(x)\sin\frac{n\pi x}{L}\,dx. \end{aligned}$$

In the classical representation, the nonseries terms would be absent.

As a final example, the representation of a continuously differentiable function on $[0, L]$ with the functions $\cos[(2n-1)\pi x/(2L)]$ is given by

$$C_0 + C_1 x + \sum_{n=1}^{\infty} C_n \cos\frac{(2n-1)\pi x}{2L} = f(x), \tag{1.61}$$

with

$$\begin{aligned} C_0 &= f(L) - f'(0)L, \\ C_1 &= f'(0), \\ \tilde{f}(x) &= f(x) - C_0 - C_1 x, \\ C_n &= \frac{1}{L}\int_{-L}^{L} \tilde{f}(x)\cos\frac{(2n-1)\pi x}{2L}\,dx. \end{aligned} \tag{1.62}$$

The equation $\boldsymbol{\nabla}^2 T = 0$ in a polar coordinate system is given by

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} = 0. \tag{1.63}$$

A separable solution to Eqn. (1.63) is given by

$$T = c_0 + c_1 \log r + c_2\theta + c_3\theta\log r + \sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty} r^n\left[A_n\cos(n\theta) + B_n\sin(n\theta)\right]. \tag{1.64}$$

The term n in Eqn. (1.64) can be replaced by λ_n, where λ_n are the roots of some characteristic equation. As in the Cartesian case, the non-series terms play a critical role in the representation of a function on a given surface. For example, on a surface $r =$ constant, if one wants to represent an even function $g(\theta)$ on the domain $\theta \in [-\theta_0, \theta_0]$ using $\cos[(2n-1)\pi\theta/(2\theta_0)]$, then similar to Eqn. (1.53), we have

$$C_0 + \sum_{n=1}^{\infty} C_n \cos\frac{(2n-1)\pi\theta}{2\theta_0} = g(\theta).$$

In two-dimensional problems, the most convenient way of generating the individual harmonic terms in the solutions such as those given by Eqns. (1.44) and (1.64) is to consider the real and the imaginary parts of a complex-valued function $W(z)$ where $z = x + iy = re^{i\theta}$. Thus, in Eqn. (1.44), the individual terms are generated using $\cos(kz) = \cos[k(x + iy)] = \cos(kx)\cosh(ky) + i\sin(kx)\sinh(ky)$, $\sin(kz) = \sin[k(x+iy)] = \sin(kx)\cosh(ky) + i\cos(kx)\sinh(ky)$ (or, equivalently, $\cosh(kz)$, $\sinh(kz)$ etc.), which are obtained on using the relations $\cos(ix) = \cosh x$, $\sin(ix) = i \sinh x$ etc. Similarly, in the case of the polar coordinate system, with $z = re^{i\theta}$, the individual harmonic terms are generated by considering the real and the imaginary parts of $z^n = (re^{i\theta})^n = r^n e^{in\theta} = r^n(\cos n\theta + i \sin n\theta)$, or $\log z = \log(re^{i\theta}) = \log r + i\theta$ and so on.

Now consider the equation $\boldsymbol{\nabla}^2 T = 0$ expressed in the cylindrical coordinate system (r, θ, z) as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 T}{\partial \theta^2} + \frac{\partial^2 T}{\partial z^2} = 0. \tag{1.65}$$

An axisymmetric separable solution to the Laplace equation is

$$\begin{aligned} T &= c_0 + c_1 \log r + c_2 z + c_3 z \log r \\ &+ \sum_{n=1}^{\infty} \Big\{ A_n I_0(k_{1n} r) \cos(k_{1n} z) + B_n K_0(k_{2n} r) \cos(k_{2n} z) \\ &+ C_n I_0(k_{3n} r) \sin(k_{3n} z) + D_n K_0(k_{4n} r) \sin(k_{4n} z) \\ &+ E_n J_0(k_{5n} r) \sinh(k_{5n} z) + F_n Y_0(k_{6n} r) \sinh(k_{6n} z) \\ &+ G_n J_0(k_{7n} r) \cosh(k_{7n} z) + H_n Y_0(k_{8n} r) \cosh(k_{8n} z) \Big\}, \end{aligned} \tag{1.66}$$

where J_0 and Y_0 denote Bessel functions of the first and second kind, and I_0 and K_0 denote modified Bessel functions of the first and second kind. Once again, the nonseries terms in Eqn. (1.66) correspond to a zero separation constant, and will play a key role in what follows.

If the problem is not axisymmetric, then in place of Eqn. (1.66), we get

$$\begin{aligned} T &= c_0 + c_1 \log r + c_2 z + c_3 z \log r + c_4 \theta + c_5 \theta \log r + c_6 \theta z + c_7 \theta z \log r \\ &+ \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \Big\{ A_{mn} J_m(k_{1n} r) \cosh(k_{1n} z) \cos m\theta + B_{mn} J_m(k_{2n} r) \cosh(k_{2n} z) \sin m\theta \\ &+ C_{mn} J_m(k_{3n} r) \sinh(k_{3n} z) \cos m\theta + D_{mn} J_m(k_{4n} r) \sinh(k_{4n} z) \sin m\theta \\ &+ E_{mn} Y_m(k_{5n} r) \cosh(k_{5n} z) \cos m\theta + F_{mn} Y_m(k_{6n} r) \cosh(k_{6n} z) \sin m\theta \\ &+ G_{mn} Y_m(k_{7n} r) \sinh(k_{7n} z) \cos m\theta + H_{mn} Y_m(k_{8n} r) \sinh(k_{8n} z) \sin m\theta \Big\}, \end{aligned} \tag{1.67}$$

with similar terms where J is replaced by I, Y is replaced by K, cosh is replaced by cos, and sinh is replaced by sin appended to the above solution. As usual, m in Eqn. (1.67) can be replaced by λ_m, where λ_m are, say, the roots of a characteristic equation.

If the domain is a solid cylinder and the boundary conditions are axisymmetric, then, based on Eqn. (1.66), we represent a function at a surface $z =$ constant as

$$c_0 + \sum_{n=1}^{\infty} A_n J_0\left(\frac{\lambda_n r}{R}\right) = f(r), \tag{1.68}$$

where λ_n, $n = 1, 2, \ldots, \infty$, are, for example, the positive roots of say $J_0(x) = 0$ or $J_1(x) = 0$.

If they are the roots of $J_0(x) = 0$, then using Eqns. (1.6) and (1.68), we have

$$
\begin{aligned}
c_0 &= f(R), \\
\tilde{f}(r) &:= f(r) - f(R), \\
A_n &= \frac{2}{R^2 J_1^2(\lambda_n)} \int_0^R \hat{r}\tilde{f}(\hat{r}) J_0\left(\frac{\lambda_n \hat{r}}{R}\right) d\hat{r}.
\end{aligned}
$$

If, however, λ_n, $n = 1, 2, \ldots, \infty$, are the positive roots of $J_1(x) = 0$, then the c_0 term is present even in the classical treatments and from Eqns. (1.7) and (1.68), we have

$$
\begin{aligned}
c_0 &= \frac{2}{R^2} \int_0^R \hat{r} f(\hat{r})\, d\hat{r}, \\
A_n &= \frac{2}{R^2 J_0^2(\lambda_n)} \int_0^R \hat{r} f(\hat{r}) J_0\left(\frac{\lambda_n \hat{r}}{R}\right) d\hat{r}.
\end{aligned}
$$

Finally, consider the equation $\boldsymbol{\nabla}^2 T = 0$ expressed in the spherical coordinate system (r, θ, ϕ) as

$$
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial T}{\partial r}\right) + \frac{1}{r^2}\frac{\partial}{\partial \xi}\left((1-\xi^2)\frac{\partial T}{\partial \xi}\right) + \frac{1}{r^2(1-\xi^2)}\frac{\partial^2 T}{\partial \phi^2} = 0, \tag{1.69}
$$

where $\xi := \cos\theta \in [-1, 1]$. The most general axisymmetric separable solution to Eqn. (1.69) in terms of (r, ξ) is

$$
T = \sum_{n=0}^{\infty} \left[A_n r^n + B_n r^{-(n+1)}\right] \left[C_n P_n(\xi) + D_n Q_n(\xi)\right], \tag{1.70}
$$

where $P_n(\xi)$ are Legendre polynomials. The Legendre functions of the second kind $Q_n(\xi)$ are singular at $\xi = \pm 1$, and hence have to be excluded in case the domain is spherical or is the annular region between two hollow spheres. The solution corresponding to a zero separation constant is of the form $c_0 + c_1/r$ which is already captured by the $n = 0$ term in Eqn. (1.70), and thus unlike the ordinary Fourier and Fourier–Bessel series that we considered in the preceding sections, there are no missing terms in the case of a Fourier-Legendre series.

The most general separable solution of Eqn. (1.69) (which is a generalization of Eqn. (1.70) to the non-axisymmetric case is

$$
T = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \left[A_n r^n + B_n r^{-(n+1)}\right] \left[C_n P_n^{(m)}(\xi) + D_n Q_n^{(m)}(\xi)\right] \left[E_n \cos m\phi + F_n \sin m\phi\right]. \tag{1.71}
$$

As usual, the integers m and n can be replaced by the roots of appropriate characteristic equations.

From Eqn. (1.70) and by considering the boundary condition at the inner or outer radius of a spherical domain, it follows that for an arbitrary continuous function $f(\xi)$ defined on $[-1, 1]$, we have (after renaming the constant)

$$\sum_{n=0}^{\infty} A_n P_n(\xi) = f(\xi),$$

where, on using Eqn. (1.28), we have

$$A_n = \frac{2n+1}{2} \int_{-1}^{1} f(\hat{\xi}) P_n(\hat{\xi})\, d\hat{\xi}.$$

1.5.2 Separable solutions to the transient heat conduction equation $\boldsymbol{\nabla}^2 T = \frac{1}{\alpha}\frac{\partial T}{\partial t}$ with $\alpha > 0$

In the Cartesian system, a separable solution is of the form

$$T = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{lmn} \cos(\alpha_l x)\cos(\beta_m y)\cos(\gamma_n z) e^{-\alpha(\alpha_l^2+\beta_m^2+\gamma_n^2)t}, \tag{1.72}$$

and similar combinations involving $[\cos(.), \cos(.), \sin(.)]$, etc. associated with (x, y, z) which need to be added to the above solution. If there is no dependence on z, say, then γ_n can be set to zero, so that the resulting form is

$$T = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \cos(\alpha_m x)\cos(\beta_n y) e^{-\alpha(\alpha_m^2+\beta_n^2)t}, \tag{1.73}$$

and similar combinations involving $[\cos(.), \sin(.)]$, etc. associated with (x, y), which need to be added to the above solution.

In a cylindrical coordinate system, a separable solution is of the form

$$T = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{lmn} J_{\gamma_n}(\alpha_l r)\cos(\beta_m z)\cos(\gamma_n \theta) e^{-\alpha(\alpha_l^2+\beta_m^2)t}, \tag{1.74}$$

and similar combinations involving $[Y_.(.), \cos(.), \sin(.)]$, etc. which need to be added to the above solution. If there on no dependence on θ, then γ_n can be taken as zero, so that the resulting form is

$$T = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_0(\alpha_m r)\cos(\beta_n z) e^{-\alpha(\alpha_m^2+\beta_n^2)t}, \tag{1.75}$$

and similar combinations involving $[Y_{\cdot}(.), \sin(.)]$, etc. associated with (r, z) which need to be added to the above solution. For a polar coordinate system, we can take β_m to be zero, so that the resulting form is

$$T = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_{\gamma_n}(\alpha_m r)\cos(\gamma_n\theta)e^{-\alpha\alpha_m^2 t}, \tag{1.76}$$

and similar combinations involving $[Y_{\cdot}(.), \sin(.)]$, etc. associated with (r, θ) which need to be added to the above solution.

In a spherical coordinate system, a separable solution with $\xi := \cos\theta$ is of the form

$$T = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{lmn} j_{\beta_m}(\alpha_l r) P_{\beta_m}^{(\gamma_n)}(\xi)\cos(\gamma_n\phi)e^{-\alpha\alpha_l^2 t}, \tag{1.77}$$

and similar combinations involving $[y_{\cdot}(.), Q_{\cdot}^{(\cdot)}(.), \sin(.)]$, etc. associated with the variables (r, ξ, ϕ) which need to be added to the above solution. If there on no dependence on ϕ, then γ_n can be taken as zero, so that the resulting form is

$$T = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} j_{\beta_n}(\alpha_m r) P_{\beta_n}(\xi)e^{-\alpha\alpha_m^2 t}, \tag{1.78}$$

and similar combinations involving $[y_{\cdot}(.), Q_{\cdot}(.)]$, etc. associated with the variables (r, ξ) which need to be added to the above solution.

1.5.3 Separable solutions to the wave equation $\boldsymbol{\nabla}^2 p_\Delta = \frac{1}{a_0^2}\frac{\partial^2 p_\Delta}{\partial t^2}$

In the Cartesian system, a separable solution is of the form

$$p_\Delta = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{lmn}\cos(\alpha_l x)\cos(\beta_m y)\cos(\gamma_n z)\cos[a_0 t\sqrt{\alpha_l^2 + \beta_m^2 + \gamma_n^2}], \tag{1.79}$$

and similar combinations involving $[\cos(.), \cos(.), \sin(.), \sin(.)]$, etc. associated with (x, y, z, t) which need to be added to the above solution. If there is no dependence on z, say, then γ_n can be set to zero, so that the resulting form is

$$p_\Delta = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}\cos(\alpha_m x)\cos(\beta_n y)\cos[a_0 t\sqrt{\alpha_m^2 + \beta_n^2}], \tag{1.80}$$

and similar combinations involving $[\cos(.), \sin(.), \sin(.)]$, etc. associated with (x, y, t), which need to be added to the above solution.

In a cylindrical coordinate system, a separable solution is of the form

$$p_\Delta = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{lmn} J_{\gamma_n}(\alpha_l r)\cos(\beta_m z)\cos(\gamma_n\theta)\cos[a_0 t\sqrt{\alpha_l^2+\beta_m^2}], \tag{1.81}$$

and similar combinations involving $[Y_.(.), \cos(.), \sin(.), \sin(.)]$, etc. associated with the variables (r, z, θ, t) which need to be added to the above solution. If there on no dependence on θ, then γ_n can be taken as zero, so that the resulting form is

$$p_\Delta = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_0(\alpha_m r)\cos(\beta_n z)\cos[a_0 t\sqrt{\alpha_m^2+\beta_n^2}], \tag{1.82}$$

and similar combinations involving $[Y_.(.), \cos(.), \sin(.)]$, etc. associated with (r, z, t) which need to be added to the above solution. For a polar coordinate system, we can take β_m to be zero, so that the resulting form is

$$p_\Delta = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_{\gamma_n}(\alpha_m r)\cos(\gamma_n\theta)\cos(a_0\alpha_m t), \tag{1.83}$$

and similar combinations involving $[Y_.(.), \cos(.), \sin(.)]$, etc. associated with (r, θ, t) which need to be added to the above solution.

In a spherical coordinate system, a separable solution with $\xi := \cos\theta$ is of the form

$$p_\Delta = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{lmn} j_{\beta_m}(\alpha_l r) P_{\beta_m}^{(\gamma_n)}(\xi)\cos(\gamma_n\phi)\cos(\alpha_l a_0 t), \tag{1.84}$$

and similar combinations involving $[y_.(.), Q_.^{(.)}(.), \sin(.), \sin(.)]$, etc. associated with the variables (r, ξ, ϕ, t) which need to be added to the above solution. If there on no dependence on ϕ, then γ_n can be taken as zero, so that the resulting form is

$$p_\Delta = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} j_{\beta_n}(\alpha_m r) P_{\beta_n}(\xi)\cos(\alpha_m a_0 t), \tag{1.85}$$

and similar combinations involving $[y_.(.), Q_.(.), \sin(.)]$, etc. associated with the variables (r, ξ, t) which need to be added to the above solution.

1.5.4 Separable solutions to the Helmholtz equation $\boldsymbol{\nabla}^2\tilde{p}+k^2\tilde{p}=0$

Assuming k is prescribed[1], in the Cartesian system, with $\alpha_m^2+\beta_n^2+\gamma_{mn}^2=k^2$, a separable solution is of the form

$$\tilde{p}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\cos(\alpha_m x)\cos(\beta_n y)\cos(\gamma_{mn}z), \tag{1.87}$$

and similar combinations involving $[\cos(.),\cos(.),\sin(.)]$, etc. associated with (x,y,z) which need to be added to the above solution.

If there is no dependence on y, say, then β_n can be set to zero, so that with $\alpha_m^2+\gamma_m^2=k^2$, the resulting form is

$$\tilde{p}=\sum_{m=1}^{\infty}A_m\cos(\alpha_m x)\cos(\gamma_m z), \tag{1.88}$$

and similar combinations involving $[\cos(.),\sin(.)]$, etc. associated with (x,z) which need to be added to the above solution.

If there is no dependence on y and z, then the solution reduces to

$$\tilde{p}=c_1\cos kx+c_2\sin kx. \tag{1.89}$$

Note that the solution in Eqn. (1.89) and the solution in Eqn. (1.88) needs to be added to the solution in Eqn. (1.86) while dealing with three-dimensional problems (similarly, the solution is Eqn. (1.89) needs to be added to the one in Eqn. (1.88) while dealing with two-dimensional problems).

In a cylindrical coordinate system, with $\alpha_m^2+\beta_m^2=k^2$, a separable solution is of the form

$$\tilde{p}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}J_{\gamma_n}(\alpha_m r)\cos(\beta_m z)\cos(\gamma_n\theta), \tag{1.90}$$

and similar combinations involving $[Y_{.}(.),\cos(.),\sin(.)]$, etc. associated with the variables (r,z,θ) which need to be added to the above solution. If there on no dependence on θ, then γ_n can be taken as zero, so that with $\alpha_m^2+\beta_m^2=k^2$, the resulting form is

$$\tilde{p}=\sum_{m=1}^{\infty}A_mJ_0(\alpha_m r)\cos(\beta_m z), \tag{1.91}$$

[1]If k is not prescribed, as in the case of an eigenvalue problem then in place of Eqn. (1.87), we get

$$\tilde{p}=\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{lmn}\cos(\alpha_l x)\cos(\beta_m y)\cos(\gamma_n z). \tag{1.86}$$

A similar extra summation gets introduced in solutions to the Helmholtz equation in other coordinate systems, e.g., see Eqns. (1.92) and (1.93).

and similar combinations involving $[Y_{\cdot}(.), \sin(.)]$, etc. associated with the variables (r, z) which need to be added to the above solution. For a polar coordinate system, we can take β_m to be zero, so that the resulting form is

$$\tilde{p} = \sum_{n=1}^{\infty} A_n J_{\gamma_n}(kr) \cos(\gamma_n \theta), \tag{1.92}$$

and similar combinations involving $[Y_{\cdot}(.), \sin(.)]$, etc. associated with the variables (r, θ) which need to be added to the above solution. As usual, in the case of an eigenvalue problem, in place of Eqn. (1.92) we have

$$\tilde{p} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} J_{\gamma_n}(\lambda_m r) \cos(\gamma_n \theta). \tag{1.93}$$

In a spherical coordinate system, a separable solution with $\xi := \cos\theta$ is of the form

$$\tilde{p} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} j_{\alpha_n}(kr) P_{\alpha_n}^{(\beta_m)}(\xi) \cos(\beta_m \phi), \tag{1.94}$$

and similar combinations involving $[y_{\cdot}(.), Q_{\cdot}^{(\cdot)}(.), \sin(.)]$, etc. associated with the variables (r, ξ, ϕ) which need to be added to the above solution. As usual in the case of an eigenvalue problem, k is replaced by λ_l with an additional sum over l. If there on no dependence on ϕ, then β_n in Eqn. (1.94) can be taken as zero, so that the solution form is

$$\tilde{p} = \sum_{m=1}^{\infty} A_m j_{\alpha_m}(kr) P_{\alpha_m}(\xi), \tag{1.95}$$

and similar combinations involving $[y_{\cdot}(.), Q_{\cdot}(.)]$, etc. associated with the variables (r, ξ) which need to be added to the above solution. As mentioned above, in the case of an eigenvalue problem, in place of Eqn. (1.95) we have

$$\tilde{p} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} j_{\alpha_n}(\lambda_m r) P_{\alpha_n}(\xi). \tag{1.96}$$

Functions ϕ that satisfy the biharmonic equation $\boldsymbol{\nabla}^4\phi = 0$ are known as biharmonic functions. If ϕ is harmonic, then the functions ϕ, $x\phi$, $y\phi$, $z\phi$, $(x^2+y^2+z^2)\phi$ are biharmonic. In the two-dimensional context, if ϕ is harmonic, then the functions ϕ, $x\phi$, $y\phi$ and $(x^2+y^2)\phi$ are biharmonic. Thus, separable solutions to $\boldsymbol{\nabla}^4\phi = 0$ can be generated from the solutions to the Laplace equation using this result.

1.6 Laplace transform

The Laplace transform of a function $f(t)$ is defined by

$$\bar{f}(s) = \int_0^\infty f(t)e^{-st}\,dt.$$

The inverse Laplace transform is given by

$$f(t) = \lim_{\omega\to\infty} \frac{1}{2\pi i} \int_{c_0-i\omega}^{c_0+i\omega} \bar{f}(s)e^{st}ds, \tag{1.97}$$

where the real constant c_0 is chosen such that the line $s = c_0$ is to the *right* of any singularity of $\bar{f}(s)$ in the complex s plane. Rather than with the use of Eqn. (1.97), the inversion is usually carried out directly with the aid of tables and the use of the convolution theorem.

In order to use the tables, one needs to express the Laplace transform $\bar{f}(s)$ which is generally of the form $P(s)/Q(s)$, with $P(s)$ being of a lower degree than $Q(s)$, in the form

$$\frac{P(s)}{Q(s)} = \sum_{i=1}^{N} \sum_{k_i=1}^{m_i} \frac{c_{k_i}^{(i)}}{(s-s_i)^{k_i}}, \tag{1.98}$$

where s_i, $i = 1, 2, \ldots, N$ are the distinct roots of $Q(s)$, m_i is the multiplicity of each root s_i, and the $c_{k_i}^{(i)}$ associated with each s_i are independent of s. In order to determine these constants, we multiply both sides by $(s - s_i)^{m_i}$, successively differentiate with respect to s, and evaluate these derivatives at s_i. The final result is

$$c_{k_i}^{(i)} = \lim_{s\to s_i} \frac{1}{(m_i-k_i)!} \frac{d^{m_i-k_i}}{ds^{m_i-k_i}} \left[\frac{P(s)}{\frac{Q(s)}{(s-s_i)^{m_i}}} \right]. \tag{1.99}$$

When all the roots of $Q(s)$ are distinct, then $m_i = 1$ for all i, and in such a case the constants are given by

$$c^{(i)} = \left[\frac{P(s)}{dQ/ds} \right]_{s=s_i}. \tag{1.100}$$

Eqn. (1.100) is known as the Heaviside formula.

By the final value theorem

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} s\bar{f}(s). \tag{1.101}$$

If $f(t)$ and $g(t)$ are two functions of time, then their convolution is defined by the equation

$$f * g = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

$$= \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

From the above relations, we observe that $f * g = g * f$. The Laplace transform of the convolution $f * g$ is

$$\overline{f * g}(s) = \bar{f}(s)\bar{g}(s). \tag{1.102}$$

Thus, the inverse Laplace transform of a product of Laplace transforms of two functions is the convolution of the two functions.

Another extremely useful result is the following:

$$\mathcal{L}^{-1}[\bar{T}(s+a)] = e^{-at}T(t). \tag{1.103}$$

As an example, using Eqn. (1.103) and Table B.1 with $q := \sqrt{s/\alpha}$, and for $\alpha \geq 0$ and $z \geq 0$, we get

$$\mathcal{L}^{-1}\left(e^{-z\sqrt{q^2+\gamma^2}}\right) = \frac{z}{\sqrt{4\pi\alpha t^3}}e^{-\frac{z^2}{4\alpha t}-\alpha\gamma^2 t}, \tag{1.104a}$$

$$\mathcal{L}^{-1}\left(\frac{e^{-z\sqrt{q^2+\gamma^2}}}{\sqrt{q^2+\gamma^2}}\right) = \sqrt{\frac{\alpha}{\pi t}}e^{-\frac{z^2}{4\alpha t}-\alpha\gamma^2 t}. \tag{1.104b}$$

If $f'(t)$ denotes the derivative of a function $f(t)$, then the Laplace transform of $f'(t)$ is given by

$$\mathcal{L}[f'(t)] = \int_0^\infty e^{-st}f'(t)\,dt = [e^{-st}f(t)]_0^\infty + \int_0^\infty se^{-st}f(t)\,dt = s\mathcal{L}[f(t)] - f(0). \tag{1.105}$$

Similarly, for the second derivative, we have

$$\mathcal{L}[f''(t)] = s^2\mathcal{L}[f(t)] - sf(0) - f'(0). \tag{1.106}$$

1.7 Classification of PDEs

We deal only with the classification of second-order partial differential equation for a scalar function $u(x,t)$. Consider the partial differential equation

$$Au_{tt} + Bu_{tx} + Cu_{xx} + Du_t + Eu_x + Fu - G = 0, \tag{1.107}$$

where the coefficient A, B, C, D, E, F and G can be functions of (x,t). The key quantity that determines the type of the partial differential equation is the discriminant

$$\Delta = B^2 - 4AC.$$

At a given (x,t), the partial differential equation in Eqn. (1.107) is called

- *hyperbolic* if $\Delta(x,t) > 0$,
- *parabolic* if $\Delta(x,t) = 0$, (with the constraint $A^2 + B^2 + C^2 \neq 0$),
- *elliptic* if $\Delta(x,t) < 0$.

As examples, we have

- The wave equation $u_{tt} - u_{xx} = 0$ has discriminant $\Delta = 4$, and is hyperbolic.
- The heat equation $u_t - u_{xx} = 0$ has discriminant $\Delta = 0$, and is parabolic.
- The Poisson equation $u_{xx}+u_{yy} = 0$ (with y playing the role of t here) has discriminant $\Delta = -4$, and is elliptic.

Most of the partial differential equations that occurs in mechanics problems (e.g., the Navier–Stokes equations in fluid mechanics) are far more complicated than the prototype given by Eqn. (1.107). However, under certain conditions, approximations can be made which fall into one of the above categories. For example, the acoustic wave equation which is a low Mach number approximation of the Navier–Stokes equations is hyperbolic.

We now discuss the boundary and initial conditions to be imposed, the uniqueness and other properties of the solution to the three types of partial differential equations mentioned above. We use the steady-state heat conduction, the transient heat conduction equation and the acoustic wave equation as the prototypes for the discussion on elliptic, parabolic and hyperbolic partial differential equations, respectively.

1.7.1 Elliptic and parabolic partial differential equations

With V denoting the domain, S denoting the boundary of the domain, and T denoting the temperature, consider the heat conduction equation given by

$$\boldsymbol{\nabla}^2 T + \frac{\rho Q}{k} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \text{ on } V, \tag{1.108}$$

where $\alpha = k/(\rho c)$ is the thermal diffusivity, k is the conductivity (which is a positive constant), c is the specific heat, and Q is the heat input.

The appropriate boundary conditions are that the temperature be prescribed on part of the boundary, say S_T, and the normal heat flux be prescribed on the remaining part of the boundary, say S_q. Thus, with $S_T \cup S_q \equiv S$, we have

$$T = \bar{T} \text{ on } S_T,$$
$$(\boldsymbol{\nabla} T)\cdot \boldsymbol{n} = \bar{q} \text{ on } S_q.$$

The initial condition is given by $T|_{t=0} = T_0(\boldsymbol{x})$. The different types of boundary value problems are

- $S = S_T$: Dirichlet or temperature boundary value problem.
- $S = S_q$: Neumann or normal flux boundary value problem.
- $S = S_T \cup S_q$: Mixed boundary value problem.

The 'steady-state' version of Eqn. (1.108) is obtained by setting $\partial T/\partial t$ to zero, i.e.,

$$k\boldsymbol{\nabla}^2 T + \rho Q = 0 \text{ on } V. \tag{1.109}$$

Eqn. (1.109) is an example of an elliptic partial differential equation. Note that a steady-state solution exists if and only if

$$\begin{aligned} &Q \text{ in } V \text{ and } \boldsymbol{q}\cdot\boldsymbol{n} \text{ on } S \text{ are independent of } t, \text{ and, in addition} \\ &\int_V \rho Q \, dV - \int_S \boldsymbol{q}\cdot\boldsymbol{n}\, dS = 0. \end{aligned} \tag{1.110}$$

Needless to say, for the steady-state solution to exist, the series or integrals in the presented solution obtained have to converge. If the domain is bounded, and if S_T is not a null set, then one need not check the condition on the second line of Eqn. (1.110), since the normal flux on S_T will adjust itself so that this condition is satisfied.

However, Eqn. (1.110) has to be applied with great care on unbounded domains. For two-dimensional and some three-dimensional unbounded domains (which we do not elaborate upon in this course), if S_f denotes the union of all the surfaces other than the one at 'infinity', then the following necessary and sufficient condition ensures that a steady-state solution exists for the temperature and the flux fields:

$$\int_V \rho Q \, dV - \int_{S_f} \boldsymbol{q}\cdot\boldsymbol{n}\, dS = 0, \quad \text{(2D and some 3D problems).} \tag{1.111}$$

As an example, consider the domain to be the region outside a circular hole in an unbounded domain (so that S_f is the boundary of this circular hole), and let $Q = 0$. If a constant normal flux q_0 is applied at the boundary of the circular hole, then by virtue of Eqn. (1.111), no steady state solution exists! However, if $-k(\partial T/\partial r)_{r=a} = g(\theta)$, and if $\int_0^{2\pi} g(\hat{\theta})\, d\hat{\theta} = 0$, then a steady state solution exists. *The important point to note here is that one should first ensure that a steady-state solution exists before attempting to find one.*

In the absence of net heat input into the domain, and in the absence of net overall flux at the boundary, the steady-state temperature is the average of the initial temperature distribution. To see this, if we integrate the governing equation

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \boldsymbol{\nabla}^2 T + \frac{\rho Q}{k},$$

over the domain V, then we get

$$\frac{1}{\alpha}\frac{d}{dt}\int_V T\,dV = -\frac{1}{k}\int_S \boldsymbol{q}\cdot\boldsymbol{n}\,dS + \frac{\rho}{k}\int_V Q\,dV,$$

If the heat input Q within the domain V, and the normal flux $\boldsymbol{q}\cdot\boldsymbol{n}$ on the boundary S are such that the right-hand side of the above equation is zero, then we get $\int_V T\,dV = \text{constant}$. If the initial temperature is $f(\boldsymbol{x})$, then evaluating the constant at $t=0$, we get

$$\int_V T\,dV = \int_V f(\boldsymbol{x})\,dV, \tag{1.112}$$

at all instants of time. In particular, if the temperature reaches to a constant steady-state value, say T_f, then from Eqn. (1.112), we get

$$T_f = \frac{\int_V f(\boldsymbol{x})\,dV}{V}. \tag{1.113}$$

An example where Eqn. (1.112) holds is if the net heat input $\int_V Q\,dV$ is zero, and if the boundary is insulated. Similarly, if there is no heat input, and if a normal flux $\bar{\boldsymbol{q}}\cdot\boldsymbol{n}$ is prescribed over the entire boundary which is such that $\int_S \bar{\boldsymbol{q}}\cdot\boldsymbol{n}\,dS = 0$, then Eqn. (1.112) holds. If the temperature is prescribed on part of the boundary, and a zero normal flux on the remaining boundary, then the reaction flux on the part of the boundary where the temperature is prescribed renders $\int_S \boldsymbol{q}\cdot\boldsymbol{n}\,dS$ nonzero, and thus Eqn. (1.112) does not hold in such a situation.

Assuming that a steady-state solution exists, we now discuss the uniqueness of solution to Eqn. (1.109). Let $q := k(\boldsymbol{\nabla}T)\cdot\boldsymbol{n}$. By multiplying Eqn. (1.109) by T, and by carrying out an integration by parts, we get

$$\int_V k(\boldsymbol{\nabla}T)\cdot(\boldsymbol{\nabla}T)\,dV = \int_V \rho QT\,dV + \int_S qT\,dS. \tag{1.114}$$

Let $T_1(\boldsymbol{x})$ and $T_2(\boldsymbol{x})$ be two solutions to Eqn. (1.109), so that

$$k\boldsymbol{\nabla}^2 T_1 + \rho Q = 0 \text{ on } V, \tag{1.115a}$$

$$k\boldsymbol{\nabla}^2 T_2 + \rho Q = 0 \text{ on } V. \tag{1.115b}$$

Let $\tilde{T} := T_1 - T_2$.Subtracting Eqn. (1.115b) from Eqn. (1.115a), we get

$$k\boldsymbol{\nabla}^2\tilde{T} = 0. \tag{1.116}$$

Thus, Eqn. (1.116) has the same form as Eqn. (1.109) with $Q = 0$, so that on using Eqn. (1.114) with Q set to zero, and with $\tilde{q} := k(\boldsymbol{\nabla}\tilde{T})\cdot\boldsymbol{n}$, we get

$$\int_V k(\boldsymbol{\nabla}\tilde{T})\cdot(\boldsymbol{\nabla}\tilde{T})\,dV = \int_S \tilde{q}\tilde{T}\,dS. \tag{1.117}$$

On the part of the boundary S_T where the temperature is prescribed, we have $\tilde{T} = 0$, and on the remaining part of the boundary where the normal flux is prescribed, we have $\tilde{q} = 0$, so that $\int_S \tilde{q}\tilde{T}\, dS = 0$. Thus, Eqn. (1.117) reduces to

$$\int_V k(\boldsymbol{\nabla}\tilde{T}) \cdot (\boldsymbol{\nabla}\tilde{T})\, dV = 0. \tag{1.118}$$

Since the integrand is non-negative on the entire domain, we get $\boldsymbol{\nabla}\tilde{T} = 0$, so that $T_1 = T_2 + c_0$, where c_0 is a constant. In the Dirichlet and mixed boundary value problems, since the temperature is prescribed on the part of the boundary Γ_T, we get $c_0 = 0$, so that T is unique. However, in the Neumann boundary value problem, the temperature solution is unique modulo a constant.

To show the uniqueness for the transient problem is more involved. By using the same arguments as above starting from Eqn. (1.108), we now get

$$\frac{1}{2\alpha}\frac{d}{dt}\int_V \tilde{T}^2\, dV = -\int_V (\boldsymbol{\nabla}\tilde{T}) \cdot (\boldsymbol{\nabla}\tilde{T})\, dV \leq 0. \tag{1.119}$$

Since both solutions T_1 and T_2 satisfy the same initial conditions, we have $\tilde{T}\Big|_{t=0} = 0$. *Assuming* that

$$\lim_{t\to 0}\int_V \tilde{T}^2\, dV = 0, \tag{1.120}$$

when $\tilde{T}\Big|_{t=0} = 0$ (this assumption may get violated in certain pathological problems where the limit and the integral cannot be interchanged in Eqn. (1.120)), Eqns. (1.119) and (1.120) imply that $\int_V \tilde{T}^2\, dV \leq 0$. But we also have $\int_V \tilde{T}^2\, dV \geq 0$ since the integrand is nonnegative. Thus, we have $\int_V \tilde{T}^2\, dV = 0$, so that $\tilde{T} = 0$, yielding uniqueness of the solution in all three types of boundary-value problems.

The wave speed for the solution to a parabolic PDE is infinite, i.e., the disturbance due to say a surface loading in an unbounded domain is transmitted instantaneously to any point in the domain.

1.7.2 Hyperbolic partial differential equations

As a typical hyperbolic partial differential equation, we consider the acoustic wave equation given by

$$\boldsymbol{\nabla}^2 p_\Delta = \frac{1}{a_0^2}\frac{\partial^2 p_\Delta}{\partial t^2}. \tag{1.121}$$

where a_0 is the wave speed. Similar to the heat conduction problem, with $S \equiv S_p \cup S_a$, the allowable boundary conditions are

- The pressure p_Δ prescribed on part of the boundary S_p.
- The normal acceleration $\boldsymbol{\nabla} p_\Delta \cdot \boldsymbol{n}$ prescribed on the remaining part of the boundary S_a

Since Eqn. (1.121) has a second-partial derivative with respect to time, initial conditions have to be specified on p_Δ and $\dot{p}_\Delta$.

By multiplying Eqn. (1.121) by $\dot{p}_\Delta$ and integrating by parts, we get

$$\frac{d}{dt}\left[\int_V \left(\frac{\dot{p}_\Delta^2}{2a_0^2} + \frac{1}{2}\boldsymbol{\nabla} p_\Delta \cdot \boldsymbol{\nabla} p_\Delta\right) dV\right] = \int_S \dot{p}_\Delta(\boldsymbol{\nabla} p_\Delta \cdot \boldsymbol{n})\, dS. \tag{1.122}$$

Consider an interior acoustic problem where part of the surface S is radiating sound, and the remaining part of S is rigid, i.e., $\boldsymbol{\nabla} p_\Delta \cdot \boldsymbol{n} = 0$. If the part of S that is radiating sound is suddenly brought to rest, then $\int_S \dot{p}_\Delta(\boldsymbol{\nabla} p_\Delta \cdot \boldsymbol{n})\, dS$ is zero, so that the total 'energy'[2]

$$E_{\text{interior}} := \int_V \left(\frac{\dot{p}_\Delta^2}{2a_0^2} + \frac{1}{2}\boldsymbol{\nabla} p_\Delta \cdot \boldsymbol{\nabla} p_\Delta\right) dV$$

is conserved from that time onwards. For exterior acoustic problems, the total energy can be shown to be a non-increasing function of time after the excitation ceases.

Such 'energy conservation' is a feature of non-dissipative systems described by hyperbolic equations. In case there is dissipation, then the total energy is a non-increasing function of time. For example, in a hyperelastic structure, the total energy given by the sum of the kinetic and strain energy is conserved in the absence of external traction and body force loading. In the presence of damping, this total energy is a non-increasing function of time in the absence of external loading.

We now use Eqn. (1.122) to prove the uniqueness of the solution. Let $(p_\Delta)_1$ and $(p_\Delta)_2$ be two solutions of Eqn. (1.121), and let $\tilde{p} = (p_\Delta)_1 - (p_\Delta)_2$. Since $(p_\Delta)_1$ and $(p_\Delta)_2$ both satisfy Eqn. (1.121), we have

$$\boldsymbol{\nabla}^2 (p_\Delta)_1 = \frac{1}{a_0^2}\frac{\partial^2 (p_\Delta)_1}{\partial t^2}, \tag{1.123a}$$

$$\boldsymbol{\nabla}^2 (p_\Delta)_2 = \frac{1}{a_0^2}\frac{\partial^2 (p_\Delta)_2}{\partial t^2}. \tag{1.123b}$$

Subtracting Eqn. (1.123b) from Eqn. (1.123a), we see that $\tilde{p}$ also satisfies Eqn. (1.121). Thus, in place of Eqn. (1.122), we now get

$$\frac{d}{dt}\left[\int_V \left(\frac{\dot{\tilde{p}}^2}{2a_0^2} + \frac{1}{2}\boldsymbol{\nabla} \tilde{p} \cdot \boldsymbol{\nabla} \tilde{p}\right) dV\right] = \int_S \dot{\tilde{p}}(\boldsymbol{\nabla} \tilde{p} \cdot \boldsymbol{n})\, dS. \tag{1.124}$$

[2] E_{interior} does not have units of energy, but is an 'energy-like' term.

On the part of the boundary where the pressure is prescribed, $\dot{\tilde{p}} = 0$, while on the remaining part of the boundary where the normal acceleration is prescribed, $(\boldsymbol{\nabla}\tilde{p} \cdot \boldsymbol{n}) = 0$, so that the right hand side of Eqn. (1.124) vanishes, i.e.,

$$\frac{d}{dt}\left[\int_V \left(\frac{\dot{\tilde{p}}^2}{2a_0^2} + \frac{1}{2}\boldsymbol{\nabla}\tilde{p} \cdot \boldsymbol{\nabla}\tilde{p}\right) dV\right] = 0. \tag{1.125}$$

Thus

$$\int_V \left(\frac{\dot{\tilde{p}}^2}{2a_0^2} + \frac{1}{2}\boldsymbol{\nabla}\tilde{p} \cdot \boldsymbol{\nabla}\tilde{p}\right) dV = \text{constant}, \tag{1.126}$$

for all times. Since at time $t = 0$, $\tilde{p}(\boldsymbol{x}, 0) = \dot{\tilde{p}}(\boldsymbol{x}, 0) = 0$, the constant value in the above equation is zero. Because of their non-negative nature, we deduce that the individual terms in the integrand are zero, i.e., considering the first term,

$$\dot{\tilde{p}}(\boldsymbol{x}, t) = 0. \tag{1.127}$$

This implies that $\tilde{p}(\boldsymbol{x}, t) = \tilde{p}(\boldsymbol{x})$ at all times. But at $t = 0$, since both $(p_\Delta)_1$ and $(p_\Delta)_2$ satisfy the same initial condition, we have $\tilde{p}(\boldsymbol{x}) = 0$. Hence $\tilde{p}(\boldsymbol{x}, t) = 0$ for all $\boldsymbol{x}$ and t proving the uniqueness of the solution.

In contrast to parabolic problems, the wave speed for solutions to hyperbolic PDEs is *finite*, i.e., a disturbance due to say a surface loading is transmitted only a finite distance (depending on the wave speed) in a finite amount of time.

Oftentimes, one is interested in 'periodic steady-state' solutions to hyperbolic equations (especially in the field of acoustics and elastodynamics) when the loading is periodic, and there is a small amount of damping in the system that damps out the initial transients (although this damping is usually not part of the mathematical model). Note that the initial conditions play no role in finding the periodic steady-solution, and in that sense the solution procedure is very similar to that for an elliptic PDE.

Substituting $p_\Delta = \text{Re}[\tilde{p}e^{i\omega t}]$ into the wave equation given by Eqn. (1.121), the governing equations for $\tilde{p}$ reduces to the Helmholtz equation given by (with $k \equiv \omega/a_0$)

$$\boldsymbol{\nabla}^2\tilde{p} + k^2\tilde{p} = 0. \tag{1.128}$$

We have already presented the general solutions to the Helmholtz wave equation given by Eqn. (1.128) in various coordinate systems. The coefficients in these solutions are found using the boundary conditions.

We now show that the governing equation and boundary conditions governing the transverse vibrations of a membrane are almost identical to those of the two-dimensional acoustic equation. Let $w(x, y, t)$ denote the transverse displacement, σ denote the areal density (units of kg/m^2), $\tilde{T}$ denote the tension in the membrane (units of force per unit length), and $q(x, y, t)$ denote the normal force per unit area acting on the membrane. Further, let

A denotes the domain occupied by the membrane in the two-dimensional plane (say, the x-y plane in the above setup), C denote its contour, and $\boldsymbol{n}$ denote the unit in-plane normal to the contour. The governing equation for the membrane is given by

$$\tilde{T}\boldsymbol{\nabla}^2 w + q(x,y,t) = \sigma\frac{\partial^2 w}{\partial t^2}, \tag{1.129}$$

which can also be written as

$$\boldsymbol{\nabla}^2 w + \frac{q(x,y,t)}{\tilde{T}} = \frac{1}{c^2}\frac{\partial^2 w}{\partial t^2}, \tag{1.130}$$

where $c^2 = \tilde{T}/\sigma$ denotes the wave speed. The allowable boundary conditions are that either w or $(\boldsymbol{\nabla} w)\cdot\boldsymbol{n}$ be prescribed at every point on the contour C, while the appropriate initial conditions are that $w|_{t=0}$ and $(\partial w/\partial t)_{t=0}$ be prescribed as functions of (x,y).

A steady-state solution exists if and only if $q(x,y,t)$ is independent of t, and if

$$\int_C (\boldsymbol{\nabla} w)\cdot\boldsymbol{n}\,ds + \int_A \frac{q(x,y)}{\tilde{T}}\,dA = 0. \tag{1.131}$$

In case it exists, the steady-state solution is found by solving Eqn. (1.130) with the right hand side set to zero, i.e., by solving

$$\boldsymbol{\nabla}^2 w + \frac{q(x,y)}{T} = 0, \tag{1.132}$$

which is exactly analogous to the steady-state heat conduction equation. Thus, steady-state solutions to the membrane problem for various domains can be obtained by making the substitutions

$$\begin{aligned} &T \to w,\\ &\rho Q \to q,\\ &k \to \tilde{T}, \end{aligned}$$

On the other hand, the transient equation given by Eqn. (1.130) is exactly analogous to the acoustic wave equation. Thus, the transient solutions for membranes can be obtained by making the substitutions

$$\begin{aligned} &\tilde{p} \text{ or } p_\Delta \to w,\\ &a_0 \to c,\\ &G \to \frac{q}{\tilde{T}}. \end{aligned}$$

Chapter 2

Elliptic and parabolic partial differential equations

In this chapter, we consider the solution of elliptic and parabolic partial differential equations on various domains.

2.1 Rectangular domain

Consider the rectangular domain $[0, L] \times [0, H]$ (see Fig. 2.1). First we consider the solution of steady-state problems for which the governing equation is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\rho Q}{k} = 0. \tag{2.1}$$

Initially, we consider the case when $Q = 0$.

If the boundary conditions are given by

$$\begin{aligned} T\big|_{y=0} &= f_b(x), \quad T\big|_{y=H} = f_t(x), \\ T\big|_{x=0} &= g_l(y), \quad T\big|_{x=L} = g_r(y), \end{aligned} \tag{2.2}$$

with the constraints $f_b(0) = g_l(0)$, $f_b(L) = g_r(0)$, $f_t(0) = g_l(H)$ and $f_t(L) = g_r(H)$, then assuming the form of the solution to be given by

$$\begin{aligned} T = c_0 + c_1 x + c_2 y + c_3 xy + \sum_{n=1}^{\infty} \frac{\sin(\alpha_n x)\,[A_n \sinh[\alpha_n(H-y)] + B_n \sinh(\alpha_n y)]}{\sinh(\alpha_n H)} \\ + \sum_{n=1}^{\infty} \frac{\sin(\beta_n y)\,[C_n \sinh[\beta_n(L-x)] + D_n \sinh(\beta_n x)]}{\sinh(\beta_n L)}, \end{aligned} \tag{2.3}$$

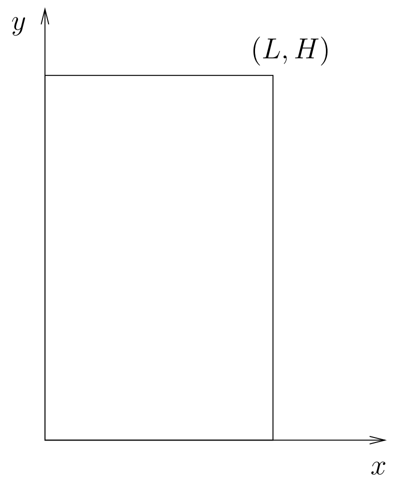

Fig. 2.1: Rectangular domain.

where $\alpha_n = n\pi/L$ and $\beta_n = n\pi/H$, we get on imposing the boundary conditions,

$$c_0 + c_1 x + \sum_{n=1}^{\infty} A_n \sin(\alpha_n x) = f_b(x),$$

$$c_0 + c_1 x + c_2 H + c_3 H x + \sum_{n=1}^{\infty} B_n \sin(\alpha_n x) = f_t(x),$$

$$c_0 + c_2 y + \sum_{n=1}^{\infty} C_n \sin(\beta_n y) = g_l(y),$$

$$c_0 + c_1 L + c_2 y + c_3 L y + \sum_{n=1}^{\infty} D_n \sin(\beta_n y) = g_r(y).$$

With the above-mentioned constraints, we get $c_0 = f_b(0) = g_l(0)$, $c_1 = [f_b(L) - f_b(0)]/L$, $c_2 = [g_l(H) - g_l(0)]/H$, and $c_3 = [g_r(H) - g_l(H) + f_b(0) - f_b(L)]/(HL)$. Let

$$\tilde{f}_b(x) = f_b(x) - c_0 - c_1 x,$$

$$\tilde{f}_t(x) = f_t(x) - c_0 - c_1 x - c_2 H - c_3 H x,$$

$$\tilde{g}_l(y) = g_l(y) - c_0 - c_2 y,$$

$$\tilde{g}_r(y) = g_r(y) - c_0 - c_1 L - c_2 y - c_3 L y.$$

The remaining constants are now given by

$$A_n = \frac{2}{L} \int_0^L \tilde{f}_b(\hat{x}) \sin(\alpha_n \hat{x})\, d\hat{x},$$

$$B_n = \frac{2}{L}\int_0^L \tilde{f}_t(\hat{x})\sin(\alpha_n\hat{x})\,d\hat{x},$$
$$C_n = \frac{2}{H}\int_0^H \tilde{g}_l(\hat{y})\sin(\beta_n\hat{y})\,d\hat{y},$$
$$D_n = \frac{2}{H}\int_0^H \tilde{g}_r(\hat{y})\sin(\beta_n\hat{y})\,d\hat{y}.$$

Note that the above solution is derived under the constraint that the imposed temperature be continuous, while the solution given by Eqn. (9), Section (5.3) of [2] does not impose any such constraint.

As an example, if $f_b(x) = f_t(x) = g_l(x) = g_r(x) = T_0$, then $c_0 = T_0$ with all the remaining constants zero, so that the steady state-solution is

$$\frac{T}{T_0} = 1. \tag{2.4}$$

As another example, if $f_t(x) = g_l(y) = g_r(y) = 0$, and $f_b(x) = T_0x(L-x)/L^2$, then we get $c_0 = c_1 = c_2 = c_3 = 0$, $B_n = C_n = D_n = 0$ for all n, so that

$$\frac{T}{T_0} = \sum_{n=1}^{\infty} \frac{8\sin\frac{(2n-1)\pi x}{L}\sinh\frac{(2n-1)\pi(H-y)}{L}}{[(2n-1)\pi]^3\sinh\frac{(2n-1)\pi H}{L}}. \tag{2.5}$$

We now consider the solution of steady-state problems for which $Q \neq 0$.

If the boundary conditions are given by

$$\begin{aligned} T\big|_{y=0} &= T\big|_{y=H} = 0, \\ T\big|_{x=0} &= T\big|_{x=L} = 0, \end{aligned} \tag{2.6}$$

then we assume the solution forms to be (these forms are the same as those that appear in the solution of the Helmholtz equation)

$$T = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\sin\alpha_m x\sin\beta_n y, \tag{2.7a}$$
$$\frac{\rho Q}{k} = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} Z_{mn}\sin\alpha_m x\sin\beta_n y, \tag{2.7b}$$

where $\alpha_m = m\pi/L$, $\beta_n = n\pi/H$, and

$$Z_{mn} = \frac{4\rho}{kLH}\int_0^H\int_0^L Q(x,y)\sin\alpha_m x\sin\beta_n y\,dx\,dy.$$

Substituting the forms given by Eqns. (2.7) into the governing equation, we get

$$A_{mn} = \frac{Z_{mn}}{\alpha_m^2 + \beta_n^2} = \frac{4\rho}{kLH\,(\alpha_m^2 + \beta_n^2)} \int_0^H \int_0^L Q(x,y) \sin\alpha_m x \sin\beta_n y \, dx \, dy. \tag{2.8a}$$

As an example, if $Q = Q_0$, we get

$$\begin{aligned}
\frac{kA_{mn}}{\rho Q_0} &= \frac{4[1-(-1)^m][1-(-1)^n](LH)^2}{\pi^2(m\pi)(n\pi)\,[(mH)^2+(nL)^2]}, \\
\frac{kT}{\rho L^2 Q_0} &= H^2 \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{16\sin\frac{(2m-1)\pi x}{L}\sin\frac{(2n-1)\pi y}{H}}{\pi^2[(2m-1)\pi][(2n-1)\pi]\,\{[(2m-1)H]^2+[(2n-1)L]^2\}} \\
&= \frac{x(L-x)}{2L^2} - \sum_{m=1}^{\infty} \frac{4\sin\frac{(2m-1)\pi x}{L}\cosh\frac{(2m-1)\pi(H-2y)}{2L}}{[(2m-1)\pi]^3\cosh\frac{(2m-1)\pi H}{2L}}.
\end{aligned} \tag{2.9}$$

If the boundary conditions are given by

$$\begin{aligned}
k\frac{\partial T}{\partial y}\bigg|_{y=0} &= T|_{y=H} = 0, \\
k\frac{\partial T}{\partial x}\bigg|_{x=0} &= k\frac{\partial T}{\partial x}\bigg|_{x=L} = 0,
\end{aligned} \tag{2.10}$$

then we assume the solution forms to be

$$T = \sum_{n=1}^{\infty} D_n \cos\beta_n y + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\cos\alpha_m x\cos\beta_n y, \tag{2.11a}$$

$$\frac{\rho Q}{k} = \sum_{n=1}^{\infty} S_n \cos\beta_n y + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} Z_{mn}\cos\alpha_m x\cos\beta_n y, \tag{2.11b}$$

where $\alpha_m = m\pi/L$, $\beta_n = (2n-1)\pi/(2H)$, and

$$\begin{aligned}
S_n &= \frac{2\rho}{kLH}\int_0^H\int_0^L Q(x,y)\cos\beta_n y\,dx\,dy, \\
Z_{mn} &= \frac{4\rho}{kLH}\int_0^H\int_0^L Q(x,y)\cos\alpha_m x\cos\beta_n y\,dx\,dy.
\end{aligned}$$

Substituting the forms given by Eqns. (2.11) into the governing equation, we get

$$D_n = \frac{S_n}{\beta_n^2} = \frac{2\rho}{kLH\beta_n^2}\int_0^H\int_0^L Q(x,y)\cos\beta_n y\,dx\,dy, \tag{2.12a}$$

$$A_{mn} = \frac{Z_{mn}}{\alpha_m^2 + \beta_n^2} = \frac{4\rho}{kLH\left(\alpha_m^2 + \beta_n^2\right)} \int_0^H \int_0^L Q(x,y) \cos\alpha_m x \cos\beta_n y \, dx \, dy. \tag{2.12b}$$

As an example, if $Q = Q_0$, then we have with $\eta := y/H$,

$$\frac{kD_n}{\rho H^2 Q_0} = \frac{4\rho(-1)^{n+1}}{k\pi(2n-1)}, \tag{2.13}$$

$$\frac{kT}{\rho H^2 Q_0} = \sum_{n=1}^{\infty} \frac{16(-1)^{n+1}\cos[(2n-1)\pi y/(2H)]}{[(2n-1)\pi]^3} = \frac{1-\eta^2}{2}. \tag{2.14}$$

Now we turn to the solution of transient problems.

If the heat input is zero, and the initial temperature is given by $f(x,y)$ with the boundary conditions given by

$$\begin{aligned} k\frac{\partial T}{\partial x}\bigg|_{x=0} &= k\frac{\partial T}{\partial x}\bigg|_{x=L} = 0, \\ k\frac{\partial T}{\partial y}\bigg|_{y=0} &= 0, \quad k\frac{\partial T}{\partial y}\bigg|_{y=H} = 0, \end{aligned} \tag{2.15}$$

then with $\alpha_m := m\pi/H$ and $\beta_n = n\pi/L$, the solution form given by

$$T = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{mn} \cos\beta_n x \cos\alpha_m y e^{-\alpha(\alpha_m^2+\beta_n^2)t}, \tag{2.16}$$

automatically satisfies the governing equation and boundary conditions. Imposing the initial condition, we get

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{mn} \cos\beta_n x \cos\alpha_m y = f(x,y),$$

which yields

$$\begin{aligned} A_{00} &= \frac{1}{LH}\int_0^H \int_0^L f(x,y)\, dx\, dy, \\ A_{m0} &= \frac{2}{LH}\int_0^H \int_0^L f(x,y)\cos\alpha_m y\, dx\, dy, \quad (m \geq 1), \\ A_{0n} &= \frac{2}{LH}\int_0^H \int_0^L f(x,y)\cos\beta_n x\, dx\, dy, \quad (n \geq 1), \\ A_{mn} &= \frac{4}{LH}\int_0^H \int_0^L f(x,y)\cos\beta_n x \cos\alpha_m y\, dx\, dy, \quad (m \geq 1, n \geq 1). \end{aligned}$$

As an example, if $f(x,y) = c_0$, then all the constants in Eqn. (2.16) except $A_{00} = c_0$ are zero, so that $T/c_0 = 1$. As another example, if $f(x,y) = c_0\cos(\pi x/L)$, then all the constants in Eqn. (2.16) except $A_{01} = c_0$ are zero, so that $T/c_0 = \cos(\pi x/L)e^{-\alpha\pi^2 t/L^2}$.

If the heat input $Q(x,y,t)$ is nonzero, the initial conditions are zero, and the boundary conditions are still given by Eqn. (2.15), then we assume the solution forms to be given by

$$\begin{aligned} T(x,y,t) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} A_{mn}(t)\cos\beta_n x\cos\alpha_m y e^{-\alpha(\alpha_m^2+\beta_n^2)t}, \\ \frac{\rho Q(x,y,t)}{k} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} Z_{mn}(t)\cos\beta_n x\cos\alpha_m y, \end{aligned} \tag{2.17}$$

where

$$\begin{aligned} Z_{00}(t) &= \frac{\rho}{kLH}\int_0^H\int_0^L Q(x,y,t)\,dx\,dy, \\ Z_{m0}(t) &= \frac{2\rho}{kLH}\int_0^H\int_0^L Q(x,y,t)\cos\alpha_m y\,dx\,dy, \quad (m\geq 1), \\ Z_{0n}(t) &= \frac{2\rho}{kLH}\int_0^H\int_0^L Q(x,y,t)\cos\beta_n x\,dx\,dy, \quad (n\geq 1), \\ Z_{mn}(t) &= \frac{4\rho}{kLH}\int_0^H\int_0^L Q(x,y,t)\cos\beta_n x\cos\alpha_m y\,dx\,dy, \quad (m\geq 1, n\geq 1). \end{aligned}$$

Substituting Eqn. (2.17) into the governing differential equation for T, we get

$$A'_{mn}(t) = \alpha Z_{mn}(t)e^{\alpha(\alpha_m^2+\beta_n^2)t},$$

which, on integrating subject to the initial condition $T|_{t=0} = 0$, yields

$$\begin{aligned} A_{00}(t) &= \alpha\int_0^t Z_{00}(\tau)\,d\tau, \\ A_{mn}(t) &= \alpha\int_0^t Z_{mn}(\tau)e^{\alpha(\alpha_m^2+\beta_n^2)\tau}\,d\tau, \quad (m,\ n,\ \text{not simultaneously zero}). \end{aligned}$$

If $Q(x,y,t)$ is a function of time alone denoted by $Q(t)$, then $Z_{00}(t) = \rho Q(t)/k$ is the only nonzero component, so that $T(x,y,t) = (\alpha\rho/k)\int_0^t Q(\tau)\,d\tau$. In particular, if $Q(x,y,t) = Q_0$, where Q_0 is a constant, we get $T(x,y,t)/Q_0 = \alpha\rho t/k$.

2.2 Doubly infinite strip

Consider the domain to be the region $-\infty < x < \infty$ and $0 \leq y \leq d$ (see Fig. 2.2). In this section, we consider only the case where there is no variation of the boundary conditions or

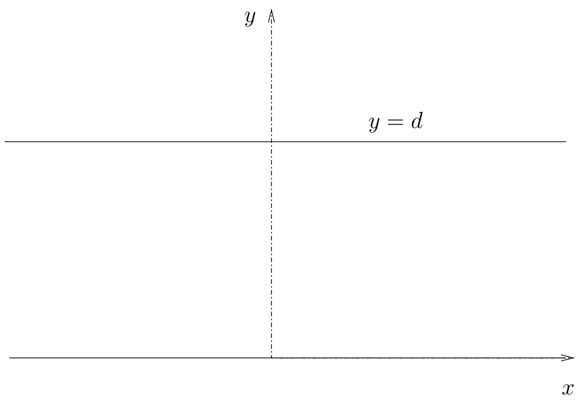

Fig. 2.2: Region bounded by two parallel planes.

the heat input with x. However, we shall indicate in the next section (see Eqn. (2.72)) how the results of this section can be extended to the case where such a variation is present. Since there is no variation along x, the governing equation is given by

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial y^2} + \frac{\rho Q(y,t)}{k}. \tag{2.18}$$

First we consider the solution of steady-state problems. Initially consider the case where $Q = 0$. If the boundary conditions are $T|_{y=0} = T_a$ and $T|_{y=d} = T_b$ then the solution is given by

$$T = \frac{(d-y)T_a + yT_b}{d}. \tag{2.19}$$

If the boundary condition is $T|_{y=0} = T_0$ and $k(\partial T/\partial y)_{y=d} = q_0$, then the steady-state temperature is

$$T = T_0 + \frac{q_0 y}{k}. \tag{2.20}$$

If the boundary conditions are $-k(\partial T/\partial y)_{y=0} = q_a$ and $k(\partial T/\partial y)_{y=d} = q_b$, then by Eqn. (1.111), a steady state solution exists if and only if $q_a + q_b = 0$. Under this constraint the solution is given by

$$T = C_0 - \frac{q_a y}{k} = C_0 + \frac{q_b y}{k}, \tag{2.21}$$

where C_0 is an arbitrary constant.

Now consider the case where $Q(y) \neq 0$. If the boundary conditions are given by $T|_{y=a} = T|_{y=d} = 0$, then the solution is given by

$$T = \sum_{n=1}^{\infty} A_n \sin\frac{n\pi y}{d}, \tag{2.22a}$$

$$\frac{\rho Q}{k} = \sum_{n=1}^{\infty} Z_n \sin \frac{n\pi y}{d}, \tag{2.22b}$$

where

$$Z_n = \frac{2\rho}{kd} \int_0^d Q(\hat{y}) \sin \frac{n\pi \hat{y}}{d}\, d\hat{y},$$

Substituting Eqns. (2.22) into the steady-state governing equation, we get

$$A_n = \frac{2\rho d}{k(n\pi)^2} \int_0^d Q(\hat{y}) \sin \frac{n\pi \hat{y}}{d}\, d\hat{y}.$$

As an example, if $Q(y) = Q_0 y/d$, then with $\xi := y/d$, we get

$$\frac{kT}{\rho d^2 Q_0} = \sum_{n=1}^{\infty} \frac{2(-1)^{n+1} \sin(n\pi\xi)}{(n\pi)^3} = \frac{\xi(1-\xi^2)}{6} \tag{2.23}$$

If the boundary conditions are $T|_{y=0} = k(\partial T/\partial y)_{y=d} = 0$, then the solution is given by

$$T = \sum_{n=1}^{\infty} A_n \sin \frac{(2n-1)\pi y}{2d}, \tag{2.24a}$$

$$\frac{\rho Q}{k} = \sum_{n=1}^{\infty} Z_n \sin \frac{(2n-1)\pi y}{2d}, \tag{2.24b}$$

where

$$Z_n = \frac{2\rho}{kd} \int_0^d Q(\hat{y}) \sin \frac{(2n-1)\pi \hat{y}}{d}\, d\hat{y},$$
$$A_n = \frac{8\rho d}{k[(2n-1)\pi]^2} \int_0^d Q(\hat{y}) \sin \frac{(2n-1)\pi \hat{y}}{2d}\, d\hat{y}.$$

If the boundary conditions are $k(\partial T/\partial y)_{y=0} = k(\partial T/\partial y)_{y=d} = 0$, then by Eqn. (1.111), a steady-state solution exists if and only if $\int_0^d Q(\hat{y})\, d\hat{y} = 0$. Under this constraint the solution is given by

$$T = C_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi y}{d}, \tag{2.25a}$$

$$\frac{\rho Q}{k} = \sum_{n=1}^{\infty} Z_n \cos \frac{n\pi y}{d}, \tag{2.25b}$$

where C_0 is an arbitrary constant, and

$$Z_n = \frac{2\rho}{kd}\int_0^d Q(\hat{y})\cos\frac{n\pi\hat{y}}{d}\,d\hat{y},$$
$$A_n = \frac{2\rho d}{k(n\pi)^2}\int_0^d Q(\hat{y})\cos\frac{n\pi\hat{y}}{d}\,d\hat{y}.$$

As an example, if $Q(\hat{y}) = Q_0(1/2 - \hat{y}/d)$, then with $\xi := y/d$, we get

$$\frac{kT}{\rho d^2 Q_0} = \frac{kC_0}{\rho d^2 Q_0} + \sum_{n=1}^{\infty}\frac{4\cos[(2n-1)\pi\xi]}{[\pi(2n-1)]^4} = \frac{kC_0}{\rho d^2 Q_0} + \frac{1-6\xi^2+4\xi^3}{24}. \tag{2.26}$$

Now we turn to the solution of transient problems. We will consider the cases of nonzero initial temperature $f(y)$ and nonzero $Q(y)$ separately. Hence, first consider the case where $f(y) = Q(y) = 0$. On taking the Laplace transform of Eqn. (2.18), since the time derivative with respect to t gets transformed to an algebraic variable s by virtue of Eqn. (1.105), the partial differential equation given by Eqn. (2.18) gets transformed to the following *ordinary* differential equation:

$$\frac{d^2\bar{T}}{dy^2} + q^2\bar{T} = 0, \tag{2.27}$$

where $q = \sqrt{-s/\alpha}$. The general solution of Eqn. (2.27) is

$$\bar{T} = c_1(s)\cos qy + c_2(s)\sin qy, \tag{2.28}$$

where the constants $c_1(s)$ and $c_2(s)$ are determined using the boundary conditions. We consider the following three types of boundary conditions:

1. $T|_{y=0} = T_a(t)$ and $T|_{y=d} = T_b(t)$:

   On imposing the boundary conditions, Eqn. (2.28) yields

   $$\bar{T} = \frac{1}{\sin qd}\left[\bar{T}_a(s)\sin q(d-y) + \bar{T}_b(s)\sin qy\right]. \tag{2.29}$$

   The roots of $\sin qd = 0$ are given by $qd = n\pi$, $n = 1, 2, \ldots, \infty$. Hence, we can write the above equation as

   $$\bar{T} = \bar{T}_a(s)\sum_{n=1}^{\infty}\frac{A_n}{s + \frac{\alpha(n\pi)^2}{d^2}} + \bar{T}_b(s)\sum_{n=1}^{\infty}\frac{B_n}{s + \frac{\alpha(n\pi)^2}{d^2}}. \tag{2.30}$$

   Since $q^2 = -s/\alpha$, we have

   $$\frac{dq}{ds} = -\frac{1}{2\alpha q}. \tag{2.31}$$

Using Eqn. (1.100), we have

$$A_n = \left. \frac{\sin q(d-y)}{\left(\frac{d(\sin qd)}{dq}\right)\frac{dq}{ds}} \right|_{q=n\pi/d} = \frac{2\alpha n\pi}{d^2}\sin\frac{n\pi y}{d}, \tag{2.32a}$$

$$B_n = \left. \frac{\sin qy}{\left(\frac{d(\sin qd)}{dq}\right)\frac{dq}{ds}} \right|_{q=n\pi/d} = \frac{2(-1)^{n+1}\alpha n\pi}{d^2}\sin\frac{n\pi y}{d}. \tag{2.32b}$$

Hence, the final solution is given by

$$T(y,t) = \frac{2\alpha}{d^2}\sum_{n=1}^{\infty} n\pi \sin\frac{n\pi y}{d}\left\{\int_0^t \left[C_n T_a(t-\tau) + D_n T_b(t-\tau)\right] e^{-\frac{\alpha(n\pi)^2\tau}{d^2}}\, d\tau\right\}, \tag{2.33}$$

where

$$C_n = 1,$$
$$D_n = (-1)^{n+1}.$$

As applications of Eqn. (2.33), we have

(a) For the case where at $t = 0$, we impose a constant temperature $T_a(t) \equiv T_a$ with $T_b(t) = 0$, we get with $\xi := y/d$,

$$\frac{T(y,t)}{T_a} = 1 - \xi - \sum_{n=1}^{\infty}\frac{2\sin(n\pi\xi)}{n\pi}e^{-\frac{\alpha(n\pi)^2 t}{d^2}}, \tag{2.34}$$

which in the limit as $t \to \infty$ reduces to the steady-state solution given by Eqn. (2.19) with $T_b = 0$.

(b) If $T_a(t) = T_a e^{-\omega t}$ and $T_b(t) = 0$, then the solution with $\gamma := \sqrt{\omega d^2/\alpha}$ is

$$\frac{T(y,t)}{T_a} = \frac{\sin[\gamma(1-\xi)]e^{-\omega t}}{\sin\gamma} - \sum_{n=1}^{\infty}\frac{2n\pi\sin n\pi\xi}{(n\pi)^2-\gamma^2}e^{-\frac{\alpha(n\pi)^2 t}{d^2}}. \tag{2.35}$$

If $T_a(t) = T_a(1-e^{-\omega t})$, then superposing the solutions given by Eqns. (2.34) and (2.35), we get

$$\frac{T(y,t)}{T_a} = 1 - \xi - \frac{\sin[\gamma(1-\xi)]e^{-\omega t}}{\sin\gamma} + 2\gamma^2\sum_{n=1}^{\infty}\frac{\sin(n\pi\xi)e^{-\frac{\alpha(n\pi)^2 t}{d^2}}}{n\pi[(n\pi)^2-\gamma^2]}. \tag{2.36}$$

Since there is now no discontinuity in the imposed temperature at $t = 0$, one obtains a much faster convergence of the series in Eqn. (2.36) than in the individual cases given by Eqns. (2.34) and (2.35). One can also obtain the flux in a straightforward way by using a term-by-term differentiation of the solution in Eqn. (2.36) without the problems of convergence that occur at $t = 0$ in the case of a discontinuous imposed temperature (see the discussion following Eqn. (2.117e)).

(c) If $T_a(t) = T_a \cos \Omega t$ and $T_b(t) = 0$, then the solution with $k_1 := \sqrt{\Omega d^2/(2\alpha)}$ is

$$\frac{T(y,t)}{2T_a} = \frac{N_1}{D} \cos \Omega t + \frac{N_2}{D} \sin \Omega t - \sum_{n=1}^{\infty} \frac{(n\pi)^3 \sin(n\pi\xi) e^{-\frac{\alpha(n\pi)^2 t}{d^2}}}{(n\pi)^4 + 4k_1^4}, \tag{2.37}$$

while if $T_a(t) = T_a \sin \Omega t$, then

$$\frac{T(y,t)}{2T_a} = \frac{N_1}{D} \sin \Omega t - \frac{N_2}{D} \cos \Omega t + 2k_1^2 \sum_{n=1}^{\infty} \frac{n\pi \sin(n\pi\xi) e^{-\frac{\alpha(n\pi)^2 t}{d^2}}}{(n\pi)^4 + 4k_1^4} \tag{2.38}$$

where

$$\begin{aligned} N_1 &= \cosh k_1 \cosh k_1(1-\xi) \sin k_1 \sin k_1(1-\xi) \\ &\quad + \sinh k_1 \sinh k_1(1-\xi) \cos k_1 \cos k_1(1-\xi), \\ N_2 &= \sinh k_1(1-\xi) \cosh k_1 \sin k_1 \cos k_1(1-\xi) \\ &\quad - \sinh k_1 \cosh k_1(1-\xi) \sin k_1(1-\xi) \cos k_1, \\ D &= \cosh 2k_1 - \cos 2k_1, \end{aligned}$$

If $T_a = T_b = Q = 0$, and the initial temperature $f(y)$ is nonzero, we assume the solution form (obtained by using separation of variables) to be

$$T(y,t) = \sum_{n=1}^{\infty} A_n \sin \frac{n\pi y}{d} e^{-\frac{\alpha(n\pi)^2 t}{d^2}}. \tag{2.39}$$

Note that this form automatically satisfies the homogeneous governing equation and the homogeneous boundary conditions at the two boundaries $y = 0$ and $y = d$. Evaluating at $t = 0$, we get

$$\sum_{n=1}^{\infty} A_n \sin \frac{n\pi y}{d} = f(y),$$

which yields

$$A_n = \frac{2}{d} \int_0^d f(y) \sin \frac{n\pi y}{d} \, dy. \tag{2.40}$$

If $f(y) = T_a = T_b = 0$, and Q is nonzero, then similar to Eqn. (2.39), we assume the solution and the heat input to be of the forms[1]

$$T(y,t) = \sum_{n=1}^{\infty} A_n(t) \sin \frac{n\pi y}{d} e^{-\frac{\alpha(n\pi)^2 t}{d^2}}, \tag{2.41a}$$

$$\frac{\rho Q(y,t)}{k} = \sum_{n=1}^{\infty} Z_n(t) \sin \frac{n\pi y}{d}, \tag{2.41b}$$

where

$$Z_n(t) = \frac{2\rho}{kd} \int_0^d Q(\hat{y},t) \sin \frac{n\pi \hat{y}}{d}\, d\hat{y}.$$

Substituting Eqns. (2.41) into Eqn. (2.18), we get

$$A_n'(t) = \alpha e^{\frac{\alpha(n\pi)^2 t}{d^2}} Z_n(t), \tag{2.42}$$

which on solving using the fact that $f(y) = 0$ yields

$$\begin{aligned} A_n(t) &= \alpha \int_0^t Z_n(\tau) e^{\frac{\alpha(n\pi)^2 \tau}{d^2}}\, d\tau \\ &= \frac{2\rho\alpha}{kd} \int_0^t \left[\int_0^d Q(\hat{y},\tau) \sin \frac{n\pi \hat{y}}{d}\, d\hat{y} \right] e^{\frac{\alpha(n\pi)^2 \tau}{d^2}}\, d\tau. \end{aligned}$$

Substituting this expression into Eqn. (2.41a), we get

$$T(y,t) = \frac{2\rho\alpha}{kd} \sum_{n=1}^{\infty} \sin \frac{n\pi y}{d} \int_0^t \left[\int_0^d Q(\hat{y},\tau) \sin \frac{n\pi \hat{y}}{d}\, d\hat{y} \right] e^{-\frac{\alpha(n\pi)^2 (t-\tau)}{d^2}}\, d\tau \tag{2.43a}$$

$$= \frac{2\rho\alpha}{kd} \sum_{n=1}^{\infty} \sin \frac{n\pi y}{d} \int_0^t \left[\int_0^d Q(\hat{y},t-\tau) \sin \frac{n\pi \hat{y}}{d}\, d\hat{y} \right] e^{-\frac{\alpha(n\pi)^2 \tau}{d^2}}\, d\tau. \tag{2.43b}$$

If $Q(y,t)$ is a function of time alone, i.e., $Q(y,t) = Q(t)$, then, as expected, Eqn. (2.43b) reduces to

$$T(y,t) = \frac{4\rho\alpha}{k} \sum_{n=1}^{\infty} \frac{1}{(2n-1)\pi} \sin \frac{(2n-1)\pi y}{d} \int_0^t Q(t-\tau) e^{-\frac{\alpha(2n-1)^2 \pi^2 \tau}{d^2}}\, d\tau. \tag{2.44}$$

As an example, if $Q(y,t) = Q_0 y/d$, then from Eqn. (2.43b), we get with $\xi := y/d$,

$$\frac{kT(y,t)}{\rho Q_0 d^2} = \frac{\xi(1-\xi^2)}{6} + \sum_{n=1}^{\infty} \frac{2(-1)^n \sin(n\pi\xi)}{n^3\pi^3} e^{-\frac{\alpha(n\pi)^2 t}{d^2}}, \tag{2.45}$$

which in the limit as $t \to \infty$ reduces to the steady-state solution given by Eqn. (2.23).

[1] The exponential term in Eqn. (2.41a) can be absorbed into $A_n(t)$; the only reason we write it separately is that, later on, it results in a slightly simplified expression for $A_n(t)$.

2. If the boundary conditions are of the type $T|_{y=0} = T_a(t)$ and $k(\partial T/\partial y)_{y=d} = q_b(t)$, and if $f(y) = Q = 0$, then the solution is given by

$$T(y,t) = \frac{\alpha}{d}\sum_{n=1}^{\infty} \sin\frac{(2n-1)\pi y}{2d}\int_0^t \left[C_n T_a(t-\tau) + \frac{D_n}{k} q_b(t-\tau)\right] e^{-\frac{(2n-1)^2\alpha\pi^2\tau}{4d^2}}\, d\tau, \tag{2.46}$$

where

$$C_n = \frac{(2n-1)\pi}{d},$$
$$D_n = 2(-1)^{n+1}.$$

As applications of Eqn. (2.46), we have

(a) If $T_a(t) = 0$ and $q_b(t) = q_0$, then

$$\frac{kT(y,t)}{q_0 d} = \xi + \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)^2}\sin\frac{(2n-1)\pi\xi}{2} e^{-\frac{\alpha(2n-1)^2\pi^2 t}{4d^2}}, \tag{2.47}$$

which in the limit as $t \to \infty$ reduces to the steady-state solution given by Eqn. (2.20) with $T_0 = 0$.

(b) If $T_a(t) = T_0$ and $q_b(t) = 0$, then

$$\frac{T(y,t)}{T_0} = 1 - 4\sum_{n=1}^{\infty}\frac{\sin\frac{(2n-1)\pi\xi}{2} e^{-\frac{\alpha(2n-1)^2\pi^2 t}{4d^2}}}{(2n-1)\pi}, \tag{2.48}$$

which in the limit as $t \to \infty$ reduces to the steady-state solution given by Eqn. (2.20) with $q_0 = 0$.

(c) If $T_a(t) = T_a \cos\Omega t$, with $q_b(t) = 0$, then with $k_1 := \sqrt{\Omega d^2/(2\alpha)}$, $\xi := y/d$, we get

$$\begin{aligned}\frac{T(y,t)}{T_a} &= \frac{N_1}{D}\cos\Omega t + \frac{N_2}{D}\sin\Omega t \\ &\quad - 4\sum_{n=1}^{\infty}\frac{(2n-1)^3\pi^3 \sin\frac{(2n-1)\pi\xi}{2}}{[(2n-1)\pi]^4 + 64k_1^4} e^{-\alpha(2n-1)^2\pi^2 t/(4d^2)},\end{aligned}$$

while if $T_a(t) = T_a \sin\Omega t$, then

$$\frac{T(y,t)}{T_a} = \frac{N_1}{D}\sin\Omega t - \frac{N_2}{D}\cos\Omega t$$

$$+ 32k_1^2 \sum_{n=1}^{\infty} \frac{(2n-1)\pi \sin \frac{(2n-1)\pi\xi}{2}}{[(2n-1)\pi]^4 + 64k_1^4} e^{-\alpha(2n-1)^2\pi^2 t/(4d^2)},$$

where

$$\begin{aligned} N_1 &= \cos k_1\xi \cosh k_1(2-\xi) + \cos k_1(2-\xi) \cosh k_1\xi, \\ N_2 &= \sin k_1\xi \sinh k_1(2-\xi) + \sin k_1(2-\xi) \sinh k_1\xi, \\ D &= \cosh 2k_1 + \cos 2k_1, \end{aligned}$$

(d) If $T_a(t) = T_a e^{-\omega t}$ and $q_b(t) = 0$, then with $\gamma = \sqrt{\omega d^2/\alpha}$, we get

$$\frac{T(y,t)}{T_a} = \frac{\cos[\gamma(1-\xi)]e^{-\omega t}}{\cos\gamma} - 4\pi \sum_{n=1}^{\infty} \frac{(2n-1)\sin\frac{(2n-1)\pi\xi}{2}}{[(2n-1)\pi]^2 - (2\gamma)^2} e^{-\frac{\alpha(2n-1)^2\pi^2 t}{4d^2}}, \quad (2.49)$$

while if $T_a(t) = 0$ and $q_b(t) = q_b e^{-\omega t}$, then

$$\frac{kT(y,t)}{q_b d} = \frac{\sin(\gamma\xi)e^{-\omega t}}{\gamma\cos\gamma} + 8 \sum_{n=1}^{\infty} \frac{(-1)^n \sin\frac{(2n-1)\pi\xi}{2}}{[(2n-1)\pi]^2 - (2\gamma)^2} e^{-\frac{\alpha(2n-1)^2\pi^2 t}{4d^2}}, \quad (2.50)$$

(e) If $T_a(t) = 0$ and $q_b(t) = q_0 \cos\Omega t$, then we have with $k_1 := \sqrt{\Omega d^2/(2\alpha)}$,

$$\begin{aligned} \frac{kT(y,t)}{q_0 d} &= \frac{N_1}{D}\cos\Omega t + \frac{N_2}{D}\sin\Omega t \\ &+ \sum_{n=1}^{\infty} \frac{8(-1)^n[(2n-1)\pi]^2}{[(2n-1)\pi]^4 + 64k_1^4} \sin\frac{(2n-1)\pi\xi}{2} e^{-\frac{\alpha(2n-1)^2\pi^2 t}{4d^2}}, \end{aligned}$$

while if $q_b(t) = q_0 \sin\Omega t$, we get

$$\begin{aligned} \frac{kT(y,t)}{q_0 d} &= \frac{N_1}{D}\sin\Omega t - \frac{N_2}{D}\cos\Omega t \\ &- \sum_{n=1}^{\infty} \frac{64(-1)^n k_1^2}{[(2n-1)\pi]^4 + 64k_1^4} \sin\frac{(2n-1)\pi\xi}{2} e^{-\frac{\alpha(2n-1)^2\pi^2 t}{4d^2}}, \end{aligned}$$

where

$$\begin{aligned} N_1 &= \sinh k_1 \sin k_1 \left[\cosh k_1\xi \sin k_1\xi - \sinh k_1\xi \cos k_1\xi\right] \\ &\quad + \cosh k_1 \cos k_1 \left[\cosh k_1\xi \sin k_1\xi + \sinh k_1\xi \cos k_1\xi\right] \\ N_2 &= \cosh k_1 \cos k_1 \left[\sinh k_1\xi \cos k_1\xi - \cosh k_1\xi \sin k_1\xi\right] \\ &\quad + \sinh k_1 \sin k_1 \left[\cosh k_1\xi \sin k_1\xi + \sinh k_1\xi \cos k_1\xi\right] \\ D &= k_1(\cosh 2k_1 + \cos 2k_1). \end{aligned}$$

If $T_a = q_b = Q = 0$, and $f(y)$ is nonzero, then the solution is given by

$$T(y,t) = \sum_{n=1}^{\infty} A_n \sin\frac{(2n-1)\pi y}{2d} e^{-\frac{\alpha(2n-1)^2\pi^2 t}{4d^2}}. \tag{2.51}$$

where

$$A_n = \frac{2}{d}\int_0^d f(\hat{y}) \sin\frac{(2n-1)\pi\hat{y}}{2d}\, d\hat{y}.$$

If $T_a = q_b = f(y) = 0$, and $Q(y,t)$ is nonzero, then following the derivation of Eqns. (2.43), and now assuming that

$$\begin{aligned} T(y,t) &= \sum_{n=1}^{\infty} A_n(t) \sin\frac{(2n-1)\pi y}{2d} e^{-\frac{\alpha(2n-1)^2\pi^2 t}{4d^2}}, \\ \frac{\rho Q(y,t)}{k} &= \sum_{n=1}^{\infty} Z_n(t) \sin\frac{(2n-1)\pi y}{2d}, \end{aligned}$$

we get

$$T(y,t) = \frac{2\rho\alpha}{kd}\sum_{n=1}^{\infty} \sin\frac{(2n-1)\pi y}{2d}\int_0^t \left[\int_0^d Q(\hat{y},\tau)\sin\frac{(2n-1)\pi\hat{y}}{2d}\, d\hat{y}\right] e^{-\frac{\alpha(2n-1)^2\pi^2(t-\tau)}{4d^2}}\, d\tau \tag{2.52a}$$

$$= \frac{2\rho\alpha}{kd}\sum_{n=1}^{\infty} \sin\frac{(2n-1)\pi y}{2d}\int_0^t \left[\int_0^d Q(\hat{y},t-\tau)\sin\frac{(2n-1)\pi\hat{y}}{2d}\, d\hat{y}\right] e^{-\frac{\alpha(2n-1)^2\pi^2\tau}{4d^2}}\, d\tau. \tag{2.52b}$$

If Q is a function of time alone, i.e., $Q(y,t) = Q(t)$, then we get

$$T(y,t) = \frac{4\rho\alpha}{k}\sum_{n=1}^{\infty} \frac{1}{(2n-1)\pi}\sin\frac{(2n-1)\pi y}{2d}\int_0^t Q(t-\tau) e^{-\frac{\alpha(2n-1)^2\pi^2\tau}{4d^2}}\, d\tau. \tag{2.53}$$

If Q is a function of y alone, then in the limit as $t \to \infty$, the solution given by Eqn. (2.52b) reduces to the steady-state solution given by Eqn. (2.24a).

3. Now consider the case where the boundary conditions are of the form $-k(\partial T/\partial y)_{y=0} = q_a(t)$ and $k(\partial T/\partial y)_{y=d} = q_b(t)$. If $f(y) = Q = 0$, then the Laplace-transformed solution is given by

$$\bar{T} = -\frac{\cos q(d-y)\bar{q}_a + \cos qy\bar{q}_b}{kq\sin qd}. \tag{2.54}$$

Eqn. (2.54) can be written as

$$\bar{T} = \frac{\alpha\left(\bar{q}_a + \bar{q}_b\right)}{kds} + \sum_{n=1}^{\infty}\left[\frac{\alpha_{1n}\bar{q}_a}{s + \frac{\alpha(n\pi)^2}{d^2}} + \frac{\alpha_{2n}\bar{q}_b}{s + \frac{\alpha(n\pi)^2}{d^2}}\right], \tag{2.55}$$

where the coefficient of $1/s$ in the first term on the right hand side of the above equation is obtained using Eqn. (1.99) as

$$\lim_{q\to 0}\frac{\left[\cos q(d-y)\bar{q}_a + (\cos qy)\bar{q}_b\right]\alpha q^2}{kq\sin qd} = \frac{\alpha\left(\bar{q}_a + \bar{q}_b\right)}{kd},$$

and the remaining coefficients are obtained using Eqn. (1.100) as

$$\alpha_{1n} = -\left.\frac{\cos q(d-y)}{kq\left(\frac{d(\sin qd)}{dq}\right)\frac{dq}{ds}}\right|_{q=n\pi/d} = \frac{2\alpha}{kd}\cos\frac{n\pi y}{d},$$

$$\alpha_{2n} = -\left.\frac{\cos qy}{kq\left(\frac{d(\sin qd)}{dq}\right)\frac{dq}{ds}}\right|_{q=n\pi/d} = \frac{2(-1)^n\alpha}{kd}\cos\frac{n\pi y}{d},$$

so that

$$\begin{aligned}\frac{T(y,t)kd}{\alpha} &= \int_0^t\left[q_a(\tau) + q_b(\tau)\right]\,d\tau + 2\sum_{n=1}^{\infty}\cos\frac{n\pi y}{d}\int_0^t q_a(t-\tau)e^{-\alpha(n\pi)^2\tau/d^2}\,d\tau \\ &\quad + 2\sum_{n=1}^{\infty}(-1)^n\cos\frac{n\pi y}{d}\int_0^t q_b(t-\tau)e^{-\alpha(n\pi)^2\tau/d^2}\,d\tau.\end{aligned} \tag{2.56}$$

As applications of Eqn. (2.56), we have

(a) If $q_b = 0$ and q_a is constant, then with $\xi := y/d$, we have

$$\frac{kT(y,t)}{q_a d} = \frac{\alpha t}{d^2} + \frac{\xi^2}{2} - \xi + \frac{1}{3} - 2\sum_{n=1}^{\infty}\frac{\cos(n\pi\xi)}{(n\pi)^2}e^{-\alpha(n\pi)^2 t/d^2},$$

(b) If $-q_a(t) = q_b(t) =$ constant, then

$$\frac{kT(y,t)}{q_b d} = \xi - \frac{1}{2} + 4\sum_{n=1}^{\infty}\frac{\cos[(2n-1)\pi\xi]}{[(2n-1)\pi]^2}e^{-\alpha(2n-1)^2\pi^2 t/d^2}.$$

which in the limit as $t\to\infty$ reduces to the steady-state solution (apart from a constant) given by Eqn. (2.21). Note that in the steady-state case, the arbitrary constant is undetermined, while in the transient case, there is no such undetermined constant by virtue of the necessity of having to meet the initial condition in addition to the boundary conditions.

(c) If $q_a = 0$ and $q_b(t) = q_0 e^{-\omega t}$, then with $\gamma := \sqrt{\omega d^2/\alpha}$

$$\frac{kT(y,t)}{q_0 d} = \frac{1}{\gamma^2}\left(1 - \frac{(\gamma\cos\gamma\xi)e^{-\omega t}}{\sin\gamma}\right) - \sum_{n=1}^{\infty}\frac{2(-1)^n\cos n\pi\xi e^{-\alpha(n\pi)^2 t/d^2}}{(n\pi)^2 - \gamma^2}. \quad (2.57)$$

(d) If $q_a = 0$ and $q_b(t) = q_0 \sin\omega t$, then with $k_1 := \sqrt{\omega d^2/(2\alpha)}$, we have

$$\frac{kT(y,t)}{q_0 d} = \frac{1}{2k_1^2} - \frac{N_1}{D}\cos\omega t - \frac{N_2}{D}\sin\omega t + 4k_1^2\sum_{n=1}^{\infty}\frac{(-1)^n\cos(n\pi\xi)e^{-\alpha(n\pi)^2 t/d^2}}{(n\pi)^4 + 4k_1^4}, \quad (2.58)$$

while if $q_b = q_0\cos\omega t$, we get

$$\frac{kT(y,t)}{q_0 d} = \frac{N_1}{D}\sin\omega t - \frac{N_2}{D}\cos\omega t - 2\sum_{n=1}^{\infty}\frac{(-1)^n(n\pi)^2\cos(n\pi\xi)e^{-\alpha(n\pi)^2 t/d^2}}{(n\pi)^4 + 4k_1^4}, \quad (2.59)$$

where

$$\begin{aligned} N_1 &= \cos k_1\sinh k_1\left[\cos k_1\xi\cosh k_1\xi - \sin k_1\xi\sinh k_1\xi\right] \\ &\quad + \sin k_1\cosh k_1\left[\cos k_1\xi\cosh k_1\xi + \sin k_1\xi\sinh k_1\xi\right], \\ N_2 &= \sin k_1\cosh k_1\left[\cos k_1\xi\cosh k_1\xi - \sin k_1\xi\sinh k_1\xi\right] \\ &\quad - \cos k_1\sinh k_1\left[\cos k_1\xi\cosh k_1\xi + \sin k_1\xi\sinh k_1\xi\right], \\ D &= k_1(\cos 2k_1 - \cosh 2k_1). \end{aligned}$$

Note the constant (nonintuitive) term that appears in the solution given by Eqn. (2.58) but not in Eqn. (2.59).

If $q_a = q_b = Q = 0$, and $f(y)$ is nonzero, then the solution is given by

$$T(y,t) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi y}{d} e^{-\frac{\alpha(n\pi)^2 t}{d^2}}, \quad (2.60)$$

where, on imposing the initial conditions, and using the orthogonality of the cosine functions, we have

$$\begin{aligned} A_0 &= \frac{1}{d}\int_0^d f(\hat{y})\,d\hat{y}, \\ A_n &= \frac{2}{d}\int_0^d f(\hat{y})\cos\frac{n\pi\hat{y}}{d}\,d\hat{y}, \quad n \geq 1. \end{aligned}$$

Finally, if $q_a = q_b = f(y) = 0$, and $Q(y,t)$ is nonzero, then assuming T and Q to be of the forms

$$T(y,t) = A_0(t) + \sum_{n=1}^{\infty} A_n(t) \cos \frac{n\pi y}{d} e^{-\frac{\alpha(n\pi)^2 t}{d^2}}, \tag{2.61a}$$

$$\frac{\rho Q(y,t)}{k} = Z_0(t) + \sum_{n=1}^{\infty} Z_n(t) \cos \frac{n\pi y}{d}, \tag{2.61b}$$

where

$$Z_0(t) = \frac{\rho}{kd} \int_0^d Q(\hat{y},t)\, d\hat{y},$$

$$Z_n(t) = \frac{2\rho}{kd} \int_0^d Q(\hat{y},t) \cos \frac{n\pi \hat{y}}{d}\, d\hat{y},$$

and following the derivation of Eqn. (2.43), we get

$$\begin{aligned}\frac{kdT(y,t)}{\rho\alpha} &= \int_0^t \int_0^d Q(\hat{y},\tau)\, d\hat{y}\, d\tau \\ &\quad + 2\sum_{n=1}^{\infty} \cos \frac{n\pi y}{d} \int_0^t \left[\int_0^d Q(\hat{y},\tau) \cos \frac{n\pi \hat{y}}{d}\, d\hat{y} \right] e^{-\frac{\alpha(n\pi)^2 (t-\tau)}{d^2}}\, d\tau \qquad (2.62a) \\ &= \int_0^t \int_0^d Q(\hat{y},\tau)\, d\hat{y}\, d\tau \\ &\quad + 2\sum_{n=1}^{\infty} \cos \frac{n\pi y}{d} \int_0^t \left[\int_0^d Q(\hat{y},t-\tau) \cos \frac{n\pi \hat{y}}{d}\, d\hat{y} \right] e^{-\frac{\alpha(n\pi)^2 \tau}{d^2}}\, d\tau. \qquad (2.62b)\end{aligned}$$

If $Q(y,t)$ is a function of time alone, then the second term in each of the equations (2.62) is zero.

As an example, if $Q(y,t) = Q_0 y/d$, then from Eqn. (2.62b), we get

$$\frac{kT(y,t)}{\rho d^2 Q_0} = \frac{\alpha t}{2d^2} + \frac{6\xi^2 - 1 - 4\xi^3}{24} + 4\sum_{n=1}^{\infty} \frac{\cos[(2n-1)\pi\xi]}{[(2n-1)\pi]^4} e^{-\frac{\alpha(2n-1)^2\pi^2 t}{d^2}}. \tag{2.63}$$

As another example, if $Q(y,t) = Q_0(1/2 - y/d)$, we get

$$\frac{kT}{\rho d^2 Q_0} = \frac{1 - 6\xi^2 + 4\xi^3}{24} - \sum_{n=1}^{\infty} \frac{4\cos[(2n-1)\pi\xi]}{[(2n-1)\pi]^4} e^{-\frac{\alpha(2n-1)^2\pi^2 t}{d^2}}, \tag{2.64}$$

which in the limit as $t \to \infty$ reduces to the steady-state solution given by Eqn. (2.26) (apart from the arbitrary constant C_0 in Eqn. (2.26) which remains indeterminate). Note that a steady-state solution exists in the case of Eqn. (2.64) but not in the case of Eqn. (2.63) since the constraints given by Eqn. (1.110) are met in the latter case, but not in the former.

2.3 Two-dimensional half space

We first consider the solution of steady-state problems on the two-dimensional half-space $y \geq 0$, and governed by the equation

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\rho Q(x, y, t)}{k} = 0. \tag{2.65}$$

Consider the case where $Q = 0$ and $T|_{y=0} = f(x)$. We assume the temperature form to be given by

$$T = \int_0^\infty [A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x] \, e^{-\lambda y} \, d\lambda, \tag{2.66}$$

which automatically satisfies Eqn. (2.65). Imposing the boundary condition, we get

$$\int_0^\infty [A(\lambda) \cos \lambda x + B(\lambda) \sin \lambda x] \; d\lambda = f(x),$$

which on inverting using Eqn. (1.37) yields

$$A(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\hat{x}) \cos \lambda \hat{x} \, d\hat{x}, \tag{2.67a}$$

$$B(\lambda) = \frac{1}{\pi} \int_{-\infty}^{\infty} f(\hat{x}) \sin \lambda \hat{x} \, d\hat{x}. \tag{2.67b}$$

Substituting Eqns. (2.67) into Eqns. (2.66), and carrying out the integration with respect to λ using Eqn. (A.2b), we get

$$T = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(\hat{x}) \, d\hat{x}}{[(x - \hat{x})^2 + y^2]}, \tag{2.68}$$

which agrees with the solution presented by Eqn. (20), Section (5.2) of [2] (with x and y interchanged). As usual, the boundary condition on the surface $y = 0$ is to be verified by evaluating the limit $y \to 0$ *after* evaluating the integral in Eqn. (2.68).

As an example, if $f(\hat{x}) = T_0$ for $-a \leq x \leq b$, and zero elsewhere, then we get

$$\begin{aligned}
\frac{T}{T_0} &= \frac{1}{\pi} \left[\tan^{-1} \left(\frac{b - x}{y} \right) + \tan^{-1} \left(\frac{a + x}{y} \right) \right], \\
\frac{q_x}{k} &= \frac{(a + b)(a - b + 2x) y}{\pi \left[(b - x)^2 + y^2 \right] \left[(a + x)^2 + y^2 \right]}, \\
\frac{q_y}{k} &= \frac{(a + b) \left[(b - x)(a + x) + y^2 \right]}{\pi \left[(b - x)^2 + y^2 \right] \left[(a + x)^2 + y^2 \right]},
\end{aligned} \tag{2.69}$$

which in the limit $y \to 0$ yields the imposed boundary condition on the temperature. Although we have considered a discontinuity in the prescribed temperature on the boundary, such a problem is not well-posed since it results in an infinite flux at $(x,y) = (-a,0)$ and at $(x,y) = (b,0)$, besides other anomalies such as different limiting values depending on the order in which one takes the limit $y \to 0$ and $x \to b$ (or $x \to -a$); the reason we have considered it here is that similar boundary conditions are often considered in the literature as in, e.g. [2]. For $a, b \to \infty$, we get the expected solution $T = T_0$.

If the boundary condition is given by

$$-k \left.\frac{\partial T}{\partial y}\right|_{y=0} = f(x), \tag{2.70}$$

then by Eqn. (1.111), a steady-state solution exists if and only if $\int_{-\infty}^{\infty} f(\hat{x})\, d\hat{x} = 0$. We shall not consider this case further because of this complication.

To find the steady-state temperature when $Q(x,y) \neq 0$, and $T|_{y=0} = 0$, we assume the solution forms to be

$$T = \int_0^\infty \int_0^\infty [A(\lambda,\gamma)\cos\lambda x + B(\lambda,\gamma)\sin\lambda x]\sin\gamma y\, d\gamma\, d\lambda, \tag{2.71a}$$

$$\frac{\rho Q}{k} = \int_0^\infty \int_0^\infty [R(\lambda,\gamma)\cos\lambda x + S(\lambda,\gamma)\sin\lambda x]\sin\gamma y\, d\gamma\, d\lambda, \tag{2.71b}$$

where

$$R(\lambda,\gamma) = \frac{2\rho}{\pi^2 k}\int_0^\infty \int_{-\infty}^\infty Q(\hat{x},\hat{y})\cos\lambda\hat{x}\sin\gamma\hat{y}\, d\hat{x}\, d\hat{y},$$

$$S(\lambda,\gamma) = \frac{2\rho}{\pi^2 k}\int_0^\infty \int_{-\infty}^\infty Q(\hat{x},\hat{y})\sin\lambda\hat{x}\sin\gamma\hat{y}\, d\hat{x}\, d\hat{y}.$$

Substituting the forms given by Eqns. (2.71) into the governing equation given by Eqn. (2.65), we get

$$A(\lambda,\gamma) = \frac{R(\lambda,\gamma)}{\lambda^2+\gamma^2},$$

$$B(\lambda,\gamma) = \frac{S(\lambda,\gamma)}{\lambda^2+\gamma^2}.$$

If the boundary condition is $(\partial T/\partial y)_{y=0} = 0$, then by Eqn. (1.111), a steady-state solution exists if and only if

$$\int_{-\infty}^\infty \int_0^\infty Q(\hat{x},\hat{y})\, d\hat{y}\, d\hat{x} = 0.$$

We shall not consider this case further because of this complication.

The results derived so far can be easily extended to the case of a doubly infinite strip as shown in Fig. 2.2. First consider the case when $Q = 0$, and the imposed temperature or normal flux conditions on the surfaces $y = 0$ and $y = d$ are functions of x. Then in place of Eqn. (2.66), we now have

$$T = \int_0^\infty [A(\lambda)\cos\lambda x + B(\lambda)\sin\lambda x]\,[C(\lambda)\cosh(\lambda y) + D(\lambda)\sinh(\lambda y)]\; d\lambda, \tag{2.72}$$

where the functions $A(\lambda)$, $B(\lambda)$, $C(\lambda)$ and $D(\lambda)$ are found based on the imposed boundary conditions.

Similarly, if $Q(x, y) \neq 0$, and if, for example, homogeneous temperature boundary conditions are imposed on the surface $y = 0$ and $y = d$, then in place of Eqn. (2.71), we now have with $\gamma_n := n\pi/d$,

$$T = \sum_{n=1}^{\infty} \sin(\gamma_n y) \int_0^\infty [A_n(\lambda)\cos\lambda x + B_n(\lambda)\sin\lambda x]\; d\lambda, \tag{2.73a}$$

$$\frac{\rho Q}{k} = \sum_{n=1}^{\infty} \sin(\gamma_n y) \int_0^\infty [R_n(\lambda)\cos\lambda x + S_n(\lambda)\sin\lambda x]\; d\lambda, \tag{2.73b}$$

which on substituting into the governing steady state equation yields

$$A_n(\lambda) = \frac{R_n(\lambda)}{\lambda^2 + \gamma_n^2},$$
$$B_n(\lambda) = \frac{S_n(\lambda)}{\lambda^2 + \gamma_n^2}.$$

We now turn to the solution of transient problems. Consider the halfspace $y \geq 0$ with the temperature specified as $T_b(t)$ on the bottom surface $y = 0$. Since there is no dependence of the boundary condition (and hence of T) on x, the governing equation is

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial y^2} + \frac{\rho Q(y, t)}{k}, \tag{2.74}$$

First consider the case where the initial temperature and the heat input are both zero. The Laplace-transformed solution of Eqn. (2.74) is given by

$$\frac{d^2\bar{T}}{dy^2} - \frac{s\bar{T}}{\alpha} = 0. \tag{2.75}$$

The solution to this equation is given by $\bar{T} = c_1 e^{qy} + c_2 e^{-qy}$, where $q = \sqrt{s/\alpha}$. Since $\bar{T} = 0$ as $y \to \infty$ and $\bar{T}(0, s) = \bar{T}_b(s)$, we have $c_1 = 0$, and $c_2 = \bar{T}_b(s)$, so that

$$\bar{T} = \bar{T}_b e^{-qy}. \tag{2.76}$$

Using Table B.1 and the convolution theorem, and with $\eta := y/\sqrt{\alpha t}$, we get

$$T(y,t) = \frac{y}{2\sqrt{\pi\alpha}} \int_0^t \frac{T_b(t-\tau)}{\tau^{3/2}} e^{-y^2/(4\alpha\tau)}\, d\tau \tag{2.77a}$$

$$= \frac{2}{\sqrt{\pi}} \int_{\eta/2}^{\infty} T_b\left(t - \frac{y^2}{4\alpha\hat{x}^2}\right) e^{-\hat{x}^2}\, d\hat{x}, \tag{2.77b}$$

where the last step has been obtained by making the substitution $\hat{x}^2 = y^2/(4\alpha\tau)$. The above solution agrees with the solution given by Eqn. (7), Section (12.4) of [2].

For $T_b(t) = T_0$, we get from Eqn. (2.77a),

$$\frac{T}{T_0} = \operatorname{erfc}\left(\frac{\eta}{2}\right).$$

Note that $\lim_{t\to\infty} T/T_0 = 1$, while $\lim_{y\to\infty} T/T_0 = 0$, which results in a 'conflict' as t and y both tend to infinity, the reason being that in our derivation of the transient solution, we require that the solution decay to zero as $y \to \infty$, while $T/T_0 = 1$ is an admissible steady-state solution if we merely impose the restriction that the solution remain bounded at infinity.

For $T_b(t) = T_0 \cos\Omega t$ and $T_0 \sin\Omega t$, we get with $\sigma := \sqrt{\Omega/(2\alpha)}$ (compare against Eqn. (8), Section (12.7) of [2]),

$$\frac{T(y,t)}{T_0} = e^{-\sigma y}\cos(\Omega t - \sigma y) - \frac{1}{\pi}\int_0^\infty \frac{\lambda e^{-(\Omega t)\lambda}}{1+\lambda^2} \sin(\sigma y\sqrt{2\lambda})\, d\lambda, \tag{2.78a}$$

$$\frac{T(y,t)}{T_0} = e^{-\sigma y}\sin(\Omega t - \sigma y) + \frac{1}{\pi}\int_0^\infty \frac{e^{-(\Omega t)\lambda}}{1+\lambda^2} \sin(\sigma y\sqrt{2\lambda})\, d\lambda, \tag{2.78b}$$

or, alternatively, on using Eqns. (A.2d) and (A.2e),

$$\frac{T(y,t)}{T_0} = e^{-\sigma y}\cos(\Omega t - \sigma y) - \frac{2}{\sqrt{\pi}}\int_0^{\eta/2} \cos\left(\Omega t - \frac{(\sigma y)^2}{2\xi^2}\right) e^{-\xi^2}\, d\xi, \tag{2.79a}$$

$$\frac{T(y,t)}{T_0} = e^{-\sigma y}\sin(\Omega t - \sigma y) - \frac{2}{\sqrt{\pi}}\int_0^{\eta/2} \sin\left(\Omega t - \frac{(\sigma y)^2}{2\xi^2}\right) e^{-\xi^2}\, d\xi. \tag{2.79b}$$

The second part of the solution in Eqns. (2.78) and (2.79) dies out as $t \to \infty$, and the 'periodic steady-state' solution in the two cases is given by

$$\frac{T_{\text{ss}}(t)}{T_0} = e^{-\sigma y}\cos(\Omega t - \sigma y), \tag{2.80a}$$

$$\frac{T_{\text{ss}}(t)}{T_0} = e^{-\sigma y}\sin(\Omega t - \sigma y). \tag{2.80b}$$

Finally, for $T_b(t) = T_0 e^{\Omega t}$, we get with $\sigma := 2\sqrt{\Omega\alpha}$ (which is imaginary when $\Omega < 0$),

$$\frac{T(y,t)}{T_0} = \frac{e^{\Omega t}}{2}\left[e^{y\sqrt{\Omega/\alpha}}\,\mathrm{erfc}\,\frac{y+\sigma t}{\sqrt{4\alpha t}} + e^{-y\sqrt{\Omega/\alpha}}\,\mathrm{erfc}\,\frac{y-\sigma t}{\sqrt{4\alpha t}}\right], \tag{2.81}$$

which agrees with the solution given by Eqn. (9), Section (2.5) of [2]. Alternative forms of the solutions in Eqns. (2.78) (which do not involve integrals, but which involve the imaginary number i) are obtained by replacing Ω by $i\Omega$ in Eqn. (2.81), and then taking the real and imaginary parts of the resulting solutions.

If $T_b(t) = 0$, and the initial temperature is $f(y)$, then we assume the solution form to be

$$T(y,t) = \int_0^\infty A(\lambda)\sin(\lambda y)\, e^{-\alpha\lambda^2 t}\, d\lambda. \tag{2.82}$$

Note that the above form automatically satisfies the governing equation and the homogeneous boundary condition at $y = 0$. Enforcing the initial condition, we get

$$\int_0^\infty A(\lambda)\sin(\lambda y)\, d\lambda = f(y),$$

which leads to

$$A(\lambda) = \frac{2}{\pi}\int_0^\infty f(\hat{y})\sin(\lambda\hat{y})\, d\hat{y}.$$

Substituting the above expressions for $A(\lambda)$ into Eqn. (2.82), and carrying out the integration with respect to λ using Eqn. (A.2c), we get

$$T(y,t) = \frac{1}{2\sqrt{\pi\alpha t}}\int_0^\infty f(\hat{y})\left[e^{-(y-\hat{y})^2/(4\alpha t)} - e^{-(y+\hat{y})^2/(4\alpha t)}\right] d\hat{y}. \tag{2.83}$$

For $f(y) = T_0$, from Eqns. (2.83) and (A.2h), we get

$$\frac{T(y,t)}{T_0} = \mathrm{erf}\,\frac{\eta}{2}.$$

If $T_b(t) = f(y) = 0$, and $Q(y,t)$ is nonzero, then we assume the solution forms to be

$$T = \int_0^\infty A(\lambda,t)\sin(\lambda y) e^{-\alpha\lambda^2 t}\, d\lambda, \tag{2.84a}$$

$$\frac{\rho Q}{k} = \int_0^\infty Z(\lambda,t)\sin(\lambda y)\, d\lambda, \tag{2.84b}$$

where

$$Z(\lambda,t) = \frac{2\rho}{k\pi}\int_0^\infty Q(\hat{y},t)\sin(\lambda\hat{y})\, d\hat{y}.$$

Substituting Eqns. (2.84) into Eqns. (2.74), and with a prime denoting a derivative with respect to time, we get

$$A'(\lambda,t) = \alpha e^{\alpha\lambda^2 t} Z(\lambda,t),$$

which leads to

$$A(\lambda,t) = \alpha \int_0^t e^{\alpha\lambda^2\tau} Z(\lambda,\tau)\,d\tau. \tag{2.85}$$

In place of a prescribed temperature, if we have an imposed normal flux condition given by $-k(\partial T/\partial y)_{y=0} = q_b(t)$, then in place of Eqn. (2.76), we now get

$$k\bar{T} = \frac{\sqrt{\alpha}\bar{q}_b e^{-y\sqrt{s/\alpha}}}{\sqrt{s}}. \tag{2.86}$$

Inverting the above Laplace transform using Table (B.1), we get in place of Eqn. (2.77a),

$$kT(y,t) = \sqrt{\frac{\alpha}{\pi}} \int_0^t \frac{q_b(t-\tau)}{\sqrt{\tau}} e^{-y^2/(4\alpha\tau)}\,d\tau \tag{2.87a}$$

$$= \frac{y}{\sqrt{\pi}} \int_{\eta/2}^{\infty} \frac{1}{\hat{x}^2} q_b\left(t - \frac{y^2}{4\alpha\hat{x}^2}\right) e^{-\hat{x}^2}\,d\hat{x}, \tag{2.87b}$$

where $\eta = y/\sqrt{\alpha t}$.

For $q_b(t) = q_0$, we get from Eqn. (2.87a),

$$\frac{kT}{q_0} = \sqrt{\frac{4\alpha t}{\pi}} e^{-y^2/(4\alpha t)} - y\,\mathrm{erfc}\,\frac{y}{\sqrt{4\alpha t}}, \tag{2.88}$$

which agrees with Eqn. (7), Section (2.9) of [2].

For $q_b(t) = q_0\cos\Omega t$ and $q_0\sin\Omega t$, we get with $\sigma := \sqrt{\Omega/(2\alpha)}$ (compare against Eqn. (13), Section (2.9) of [2]),

$$\frac{k\Omega T(y,t)}{\alpha\sigma q_0} = e^{-\sigma y}\left[\cos(\Omega t - \sigma y) + \sin(\Omega t - \sigma y)\right] - \frac{\sqrt{2}}{\pi} \int_0^{\infty} \frac{\sqrt{\lambda} e^{-(\Omega t)\lambda}}{1+\lambda^2} \cos(\sigma y\sqrt{2\lambda})\,d\lambda, \tag{2.89a}$$

$$\frac{k\Omega T(y,t)}{\alpha\sigma q_0} = -e^{-\sigma y}\left[\cos(\Omega t - \sigma y) + \sin(\Omega t - \sigma y)\right] + \frac{\sqrt{2}}{\pi} \int_0^{\infty} \frac{e^{-(\Omega t)\lambda}}{\sqrt{\lambda}(1+\lambda^2)} \cos(\sigma y\sqrt{2\lambda})\,d\lambda, \tag{2.89b}$$

or, alternatively,

$$\frac{k\Omega T(y,t)}{\alpha\sigma q_0} = e^{-\sigma y}\left[\cos(\Omega t - \sigma y) + \sin(\Omega t - \sigma y)\right] - \frac{2\sigma y}{\sqrt{\pi}} \int_0^{\eta/2} \frac{1}{\xi^2} \cos\left(\Omega t - \frac{(\sigma y)^2}{2\xi^2}\right) e^{-\xi^2}\,d\xi,$$

$$\frac{k\Omega T(y,t)}{\alpha\sigma q_0} = -e^{-\sigma y}\left[\cos(\Omega t-\sigma y)-\sin(\Omega t-\sigma y)\right] - \frac{2\sigma y}{\sqrt{\pi}}\int_0^{\eta/2}\frac{1}{\xi^2}\sin\left(\Omega t-\frac{(\sigma y)^2}{2\xi^2}\right)e^{-\xi^2}\,d\xi.$$

Finally, for $q_b(t) = q_0 e^{\Omega t}$, we get with $\sigma := 2\sqrt{\Omega\alpha}$,

$$\frac{kT(y,t)}{q_0} = -\sqrt{\frac{\alpha}{4\Omega}}\left[e^{\Omega t+y\sqrt{\Omega/\alpha}}\operatorname{erfc}\frac{y+\sigma t}{\sqrt{4\alpha t}} - e^{\Omega t-y\sqrt{\Omega/\alpha}}\operatorname{erfc}\frac{y-\sigma t}{\sqrt{4\alpha t}}\right]. \tag{2.90}$$

Alternative forms of the solutions in Eqns. (2.89) (which do not involve integrals, but which involve i) are obtained by replacing Ω by $i\Omega$ in Eqn. (2.90), and then taking the real and imaginary parts of the resulting solutions.

If $q_b(t) = 0$, and the initial temperature is $f(y)$, then we assume the solution form to be

$$T(y,t) = \int_0^\infty A(\lambda)\cos(\lambda y)\,e^{-\alpha\lambda^2 t}\,d\lambda. \tag{2.91}$$

Note that the above form automatically satisfies the governing equation and the homogeneous boundary condition at $y = 0$. Enforcing the initial condition, we get

$$\int_0^\infty A(\lambda)\cos(\lambda y)\,d\lambda = f(y),$$

which leads to

$$A(\lambda) = \frac{2}{\pi}\int_0^\infty f(\hat{y})\cos(\lambda\hat{y})\,d\hat{y}.$$

For $f(y) = T_0$, we get, as expected, $T/T_0 = 1$.

If $q_b(t) = f(y) = 0$, and $Q(y,t)$ is nonzero, then we assume the solution forms to be

$$T = \int_0^\infty A(\lambda,t)\cos(\lambda y)e^{-\alpha\lambda^2 t}\,d\lambda, \tag{2.92a}$$

$$\frac{\rho Q}{k} = \int_0^\infty Z(\lambda,t)\cos(\lambda y)\,d\lambda, \tag{2.92b}$$

where

$$Z(\lambda,t) = \frac{2\rho}{k\pi}\int_0^\infty Q(\hat{y},t)\cos(\lambda\hat{y})\,d\hat{y}.$$

If $Q(x,y,t)$ is a function of time alone, then we get

$$\frac{kT(y,t)}{\rho} = \alpha\int_0^t Q(\tau)\,d\tau. \tag{2.93}$$

For $Q = Q_0$, we get from Eqn. (2.93),

$$\frac{kT}{\rho Q_0} = \alpha t. \tag{2.94}$$

2.4 Semi-infinite halfspace–axisymmetric case

This section considers the axisymmetric version of the three-dimensional semi-infinite half-space problem. First we discuss the solution of steady-state problems. Let the boundary condition be given by $T|_{z=0} = f(r)$. We assume the temperature to be of the form

$$T = \int_0^\infty A(\lambda) J_0(\lambda r) e^{-\lambda z} \, d\lambda. \tag{2.95}$$

Note that this form automatically satisfies the steady-state governing equation. Imposing the aforementioned boundary condition, we get

$$\int_0^\infty A(\lambda) J_0(\lambda r) \, d\lambda = f(r),$$

which on using Eqn. (1.39) yields

$$T = \int_0^\infty \hat{r} f(\hat{r}) \left[\int_0^\infty \lambda J_0(\lambda r) J_0(\lambda \hat{r}) e^{-\lambda z} \, d\lambda \right] d\hat{r}. \tag{2.96}$$

The flux is obtained by using $\boldsymbol{q} = -k \boldsymbol{\nabla} T$.

As an example, if $T = T_0 \sqrt{1 - r^2/a^2}$ for $0 \le r \le a$, and zero elsewhere, we get

$$\frac{A(\lambda)}{T_0} = \frac{\sin(\lambda a) - \lambda a \cos(\lambda a)}{a \lambda^2},$$
$$\left. \frac{T}{T_0} \right|_{r=0} = 1 - \frac{z}{a} \sin^{-1} \frac{a}{\sqrt{a^2 + z^2}}.$$

Now consider the boundary condition to be

$$\left. -k \frac{\partial T}{\partial z} \right|_{z=0} = f(r).$$

Imposing the above boundary condition on the temperature form given by Eqn. (2.95), and using Eqn. (1.39), we get,

$$kT = \int_0^\infty \hat{r} f(\hat{r}) \left[\int_0^\infty J_0(\lambda r) J_0(\lambda \hat{r}) e^{-\lambda z} \, d\lambda \right] d\hat{r}. \tag{2.97}$$

The flux is obtained by using $\boldsymbol{q} = -k \boldsymbol{\nabla} T$.

As an example, if $f(r) = q_0 e^{-\beta r^2}$, then along the z-axis and on the plane $z = 0$, we get

$$\left. \frac{kT\sqrt{\beta}}{q_0} \right|_{r=0} = \frac{\sqrt{\pi}}{2} e^{\beta z^2} \operatorname{erfc}(z\sqrt{\beta}), \tag{2.98a}$$

$$\left.\frac{kT\sqrt{\beta}}{q_0}\right|_{z=0} = \frac{\sqrt{\pi}}{2} e^{-\beta r^2/2} I_0\left(\frac{\beta r^2}{2}\right). \tag{2.98b}$$

If $Q \neq 0$, and $T|_{z=0} = 0$, then we assume the solution forms to be

$$T = \int_0^\infty \int_0^\infty A(\lambda, \gamma) J_0(\lambda r) \sin \gamma z \, d\gamma \, d\lambda, \tag{2.99a}$$

$$\frac{\rho Q}{k} = \int_0^\infty \int_0^\infty Z(\lambda, \gamma) J_0(\lambda r) \sin \gamma z \, d\gamma \, d\lambda, \tag{2.99b}$$

where

$$Z(\lambda, \gamma) = \frac{2\rho\lambda}{k\pi} \int_0^\infty \int_0^\infty \hat{r} J_0(\lambda \hat{r}) \sin \gamma \hat{z} Q(\hat{r}, \hat{z}) \, d\hat{z} \, d\hat{r}.$$

Substituting the above forms into the steady-state governing equation, we get

$$A(\lambda, \gamma) = \frac{Z(\lambda, \gamma)}{\lambda^2 + \gamma^2}. \tag{2.100}$$

If $Q \neq 0$, and $(\partial T/\partial z)_{z=0} = 0$, then we assume the solution forms to be

$$T = \int_0^\infty \int_0^\infty A(\lambda, \gamma) J_0(\lambda r) \cos \gamma z \, d\gamma \, d\lambda, \tag{2.101a}$$

$$\frac{\rho Q}{k} = \int_0^\infty \int_0^\infty Z(\lambda, \gamma) J_0(\lambda r) \cos \gamma z \, d\gamma \, d\lambda, \tag{2.101b}$$

where

$$Z(\lambda, \gamma) = \frac{2\rho\lambda}{k\pi} \int_0^\infty \int_0^\infty \hat{r} J_0(\lambda \hat{r}) \cos \gamma \hat{z} Q(\hat{r}, \hat{z}) \, d\hat{z} \, d\hat{r},$$

$$A(\lambda, \gamma) = \frac{Z(\lambda, \gamma)}{\lambda^2 + \gamma^2}.$$

In the transient case, if we assume that the boundary conditions are a function of time alone, or, in case the initially imposed temperature is of the form $f(z, t)$, or the imposed heat input is of the form $Q(z, t)$, then the solutions are identical to the transient solutions presented in Section 2.3, with now z playing the role of y.

As in Section 2.3, the solution for an axisymmetric imposed temperature or imposed normal flux on the faces $z = 0$ and $z = d$ of an infinite slab (i.e., infinite along the radial direction), is obtained by replacing $e^{-\lambda z}$ in Eqn. (2.95) by a combination of $\cosh(\lambda z)$ and $\sinh(\lambda z)$ functions. For a nonzero Q, the solution is obtained by replacing γ by γ_n where γ_n are the roots of an appropriate characteristic equation obtained using the boundary conditions, and the integral with respect to γ is replaced by a summation.

2.5 Spherical domain

Consider the axisymmetric spherical domain problem. Let $\xi := \cos\theta$ and $\zeta := r/a$. The governing equation is given by

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial \xi}\left((1-\xi^2)\frac{\partial T}{\partial \xi}\right)+\frac{\rho Q(r,\xi,t)}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial t}. \tag{2.102}$$

First we consider the solution of steady-state problems on a spherical domain of radius a. Initially, consider the case where $Q=0$. If the boundary condition is given by $T|_{r=a}=g(\xi)$, then we assume the steady-state temperature to be of the form

$$T=\sum_{n=0}^{\infty}C_n\zeta^n P_n(\xi). \tag{2.103}$$

Imposing the boundary condition, we get

$$\sum_{n=0}^{\infty}C_n P_n(\xi)=g(\xi), \tag{2.104}$$

which on using Eqn. (1.28) leads to

$$C_n=\frac{2n+1}{2}\int_{-1}^{1}g(\xi)P_n(\xi)\,d\xi. \tag{2.105}$$

As an example, if $g(\xi)=T_0$, we get $C_0=T_0$, and $C_n=0$ for all $n\geq 1$, so that the steady-state temperature is $T=T_0$.

If the boundary condition is given by $k(\partial T/\partial r)_{r=a}=g(\xi)$, then by Eqn. (1.110), a steady-state solution exists if and only if $\int_{-1}^{1}g(\xi)\,d\xi=0$. Under this constraint, the solution is

$$T=C_0+\frac{a}{k}\sum_{n=1}^{\infty}C_n\zeta^n P_n(\xi), \tag{2.106}$$

where C_0 is an arbitrary constant, and

$$C_n=\frac{2n+1}{2n}\int_{-1}^{1}g(\xi)P_n(\xi)\,d\xi. \tag{2.107}$$

If $Q(r,\xi)\neq 0$, and the boundary condition is given by $T|_{r=a}=0$, then we assume the solution form to be given by[2]

$$T=\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}C_{mn}j_n\left(\lambda_m^{(n)}\zeta\right)P_n(\xi), \tag{2.108a}$$

[2]Since T and Q have the same form, the functions that appear in the double Fourier-type expansion for T and Q are the same as the functions that appear in the separable approximation to the Helmholtz equation as given by Eqn. (1.96).

$$\frac{\rho Q}{k} = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} Z_{mn} j_n\left(\lambda_m^{(n)}\zeta\right) P_n(\xi), \tag{2.108b}$$

where $\lambda_m^{(n)}$, $m = 1, 2, \ldots, \infty$, are the positive roots of $j_n(x) = 0$, and where on using Eqns. (1.17) and (1.28),

$$Z_{mn} = \frac{(2n+1)\rho}{ka^3 j_{n+1}^2(\lambda_m^{(n)})} \int_0^a \int_{-1}^1 \hat{r}^2 Q(\hat{r},\hat{\xi}) j_n\left(\frac{\lambda_m^{(n)}\hat{r}}{a}\right) P_n(\hat{\xi})\, d\hat{\xi}\, d\hat{r}.$$

Substituting Eqns. (2.108) into the steady-state form of Eqn. (2.102), we get

$$\begin{aligned} C_{mn} &= \frac{a^2 Z_{mn}}{(\lambda_m^{(n)})^2} \\ &= \frac{(2n+1)\rho}{ka(\lambda_m^{(n)})^2 j_{n+1}^2(\lambda_m^{(n)})} \int_0^a \int_{-1}^1 \hat{r}^2 Q(\hat{r},\hat{\xi}) j_n\left(\frac{\lambda_m^{(n)}\hat{r}}{a}\right) P_n(\hat{\xi})\, d\hat{\xi}\, d\hat{r}. \end{aligned} \tag{2.109}$$

As an example, if $Q = Q_0$, then by virtue of Eqn. (1.29), C_{m0}, $m = 1, 2, \ldots, \infty$, are the only nonzero components, so that on using Eqn. (1.15d), we get

$$\frac{kT}{\rho a^2 Q_0} = \frac{2}{\zeta}\sum_{m=1}^{\infty} \frac{(-1)^{m+1}\sin(m\pi\zeta)}{(m\pi)^3} = \frac{1-\zeta^2}{6}. \tag{2.110}$$

If the boundary condition is given by $(\partial T/\partial r)_{r=a} = 0$, then by Eqn. (1.110), a steady-state solution exists if and only if $\int_0^a \int_{-1}^1 Q(\hat{r},\hat{\xi})\hat{r}^2\, d\hat{r}\, d\hat{\xi} = 0$. We do not consider this case further because of this constraint. We now proceed to the solution of spherically symmetric transient problems.

Assuming spherical symmetry in the heat supply and boundary conditions, the governing equation is

$$\frac{1}{\alpha}\frac{\partial T}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right) + \frac{\rho Q(r,t)}{k}, \tag{2.111}$$

where $\alpha = k/(\rho c_v)$. Let the initial temperature and the heat input be zero, and let the boundary condition be given by

$$T\big|_{r=a} = T_a(t). \tag{2.112}$$

To solve Eqn. (2.111), it helps to introduce a new variable $u(r,t) := rT(r,t)$. In terms of this new variable, Eqn. (2.111) can be written as

$$\frac{1}{\alpha}\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial r^2} = 0, \tag{2.113}$$

with boundary conditions given by

$$u|_{r=a} = aT_a(t),$$
$$u|_{r=0} = 0.$$

Taking the Laplace transform of Eqn. (2.113), we get

$$\frac{d^2\bar{u}}{dr^2} + q^2\bar{u} = 0 \tag{2.114}$$

where $q = \sqrt{-s/\alpha}$. Then the homogeneous solution of the above equation that meets the boundary condition at $r = 0$ is $c_1 \sin qr$. Using the boundary condition $\bar{u}|_{r=a} = a\bar{T}_a$, we get

$$\bar{T} = \frac{\bar{T}_a \sin qr}{\zeta \sin qa}. \tag{2.115}$$

Similar to the inversion Eqn. (2.30), we get

$$T(r,t) = \sum_{n=1}^{\infty} \int_0^t D_n T_a(t-\tau) e^{-\frac{\alpha(n\pi)^2\tau}{a^2}}\, d\tau, \tag{2.116}$$

where

$$D_n = \frac{2(-1)^{n+1}\alpha n\pi \sin(n\pi\zeta)}{ar},$$

which agrees with Eqn. (3), Section (9.3) of [2].

As an example, if $Q = f(r) = 0$ and $T_a = C_1$, where C_1 is a constant, then we get

$$\frac{T}{C_1} = 1 - \frac{2}{\pi\zeta}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}\sin(n\pi\zeta)}{n} e^{-\frac{\alpha(n\pi)^2 t}{a^2}}, \qquad (r > 0), \tag{2.117a}$$

$$= 1 + 2\sum_{n=1}^{\infty}(-1)^n e^{-\frac{\alpha(n\pi)^2 t}{a^2}}, \qquad (r = 0,\ t > 0), \tag{2.117b}$$

$$= 0, \qquad (r \geq 0,\ t = 0), \tag{2.117c}$$

$$\frac{q_r a}{kC_1} = -\frac{2}{\pi\zeta}\sum_{n=1}^{\infty} \frac{(-1)^n}{n}\left[n\pi\cos(n\pi\zeta) - \frac{1}{\zeta}\sin(n\pi\zeta)\right] e^{-\frac{\alpha(n\pi)^2 t}{a^2}}, \quad (t > 0), \tag{2.117d}$$

$$= 0, \qquad (t = 0), \tag{2.117e}$$

which in the limit as $t \to \infty$ reduces to the steady-state solution given by $T = T_0$. Note that the flux as given by Eqn. (2.117d) does not tend to zero as $t \to 0$ (in fact, the series is nonconvergent at $t = 0$), and is a result of the discontinuity (with respect to time) in the applied temperature at $(r,t)=(a,0)$. If there is no discontinuity in the applied temperature field, then the flux expression obtained by a term-by-term differentiation of the temperature field is valid for all $t \geq 0$. To summarize, although the flux expressions are not presented elsewhere in this section, they can be obtained simply by a term-by-term differentiation of the temperature field with the range of validity being $t > 0$ or $t \geq 0$ depending on whether a discontinuous temperature field is applied or not. If a discontinuous flux is applied (with no discontinuous temperature field applied at some other boundary), then the range of validity of the obtained flux solution (as in the case of Eqn. (2.134c)) is $t \geq 0$.

As another example, if $Q = f(r) = 0$, and $T_a = T_0 \sin \omega t$, then the transient solution is found by substituting $T_a(t-\tau) = T_0 \sin \omega (t-\tau)$ into Eqn. (2.116). With $\gamma := \sqrt{\omega a^2/(2\alpha)}$, the complete solution for the case $T_a(t) = T_0 \sin \omega t$ is given by (compare against Eqn. (12), Section (9.3) of [2])

$$\frac{T}{T_0} = \frac{N_1}{D}\cos \omega t + \frac{N_2}{D}\sin \omega t - \frac{4\gamma^2}{\zeta}\sum_{n=1}^{\infty}\frac{(-1)^n n\pi \sin(n\pi\zeta)}{(n\pi)^4 + 4\gamma^4} e^{-\frac{\alpha(n\pi)^2 t}{a^2}}, \tag{2.118}$$

while that for the case $T_a(t) = T_0 \cos \omega t$ is given by

$$\frac{T}{T_0} = -\frac{N_1}{D}\sin \omega t + \frac{N_2}{D}\cos \omega t + \frac{2}{\zeta}\sum_{n=1}^{\infty}\frac{(-1)^n (n\pi)^3 \sin(n\pi\zeta)}{(n\pi)^4 + 4\gamma^4} e^{-\frac{\alpha(n\pi)^2 t}{a^2}}, \tag{2.119}$$

where

$$\begin{aligned}
N_1 &= 2(\cos\gamma \cosh\gamma\zeta \sin\gamma\zeta \sinh\gamma - \cos\gamma\zeta \cosh\gamma \sin\gamma \sinh\gamma\zeta), \\
N_2 &= 2(\cosh\gamma \cosh\gamma\zeta \sin\gamma \sin\gamma\zeta + \cos\gamma \cos\gamma\zeta \sinh\gamma \sinh\gamma\zeta), \\
D &= \zeta(\cosh 2\gamma - \cos 2\gamma).
\end{aligned}$$

If $T_a(t) = T_0 e^{-\omega t}$, then with $\gamma := \sqrt{\omega a^2/\alpha}$, we get

$$\frac{T}{T_0} = \frac{\sin\gamma\zeta}{\zeta \sin\gamma} e^{-\omega t} - \frac{2}{\zeta}\sum_{n=1}^{\infty}\frac{(-1)^{n+1} n\pi \sin(n\pi\zeta) e^{-\alpha(n\pi)^2 t/a^2}}{(n\pi)^2 - \gamma^2}. \tag{2.120}$$

The evolution of the initial temperature field $f(r)$ under the conditions $Q = T_a = 0$ can be found by assuming the solution form to be

$$T = \frac{1}{\zeta}\sum_{n=1}^{\infty} C_n \sin(n\pi\zeta) e^{-\frac{\alpha n^2 \pi^2 t}{a^2}},$$

where, on imposing the initial condition, we get

$$C_n = \frac{2}{a^2}\int_0^a \hat{r} f(\hat{r}) \sin\frac{n\pi\hat{r}}{a}\, d\hat{r}.$$

As an example, if $f(r) = C_0$, then we get

$$\frac{T}{C_0} = \frac{2}{\pi\zeta}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin(n\pi\zeta)}{n}e^{-\frac{\alpha(n\pi)^2 t}{a^2}}, \qquad (r > 0), \tag{2.121a}$$

$$= -2\sum_{n=1}^{\infty}(-1)^n e^{-\frac{\alpha(n\pi)^2 t}{a^2}}, \qquad (r = 0,\ t > 0), \tag{2.121b}$$

$$= 1, \qquad (r \geq 0,\ t = 0), \tag{2.121c}$$

$$\frac{q_r a}{kC_0} = \frac{2}{\pi\zeta}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[n\pi\cos(n\pi\zeta) - \frac{1}{\zeta}\sin(n\pi\zeta)\right]e^{-\frac{\alpha(n\pi)^2 t}{a^2}}, \qquad (t > 0), \tag{2.121d}$$

$$= 0, \qquad (t = 0). \tag{2.121e}$$

If $f(r) = C_0(1-\zeta)$, then we get

$$\frac{T}{C_0} = \frac{8}{\zeta}\sum_{n=1}^{\infty}\frac{\sin[(2n-1)\pi\zeta]}{[(2n-1)\pi]^3}e^{-\frac{\alpha(2n-1)^2\pi^2 t}{a^2}},$$

which agrees with Eqn. (13), Section (9.3) of [2], while if $f(r) = C_0\sin(\pi\zeta)/\zeta$, then $C_1 = C_0$ is the only nonzero constant, so that

$$\frac{T}{C_0} = \frac{\sin(\pi\zeta)e^{-\alpha\pi^2 t/a^2}}{\zeta}, \tag{2.122}$$

which (after the correction of a typographical error) agrees with Eqn. (17), Section (9.3) of [2].

Similarly, if $f(r) = T_a = 0$, but $Q(r,t)$ is nonzero, then we assume the solution forms to be

$$T = \frac{1}{\zeta}\sum_{n=1}^{\infty}C_n(t)\sin(n\pi\zeta)e^{-\frac{\alpha n^2\pi^2 t}{a^2}}, \tag{2.123a}$$

$$\frac{\rho Q}{k} = \frac{1}{\zeta}\sum_{n=1}^{\infty}Z_n(t)\sin(n\pi\zeta), \tag{2.123b}$$

where

$$Z_n(t) = \frac{2\rho}{ka^2}\int_0^a \hat{r}Q(\hat{r},t)\sin\frac{n\pi\hat{r}}{a}\,d\hat{r}.$$

Substituting Eqns. (2.123) into Eqn. (2.111), we get

$$C_n'(t) = \alpha Z_n(t)e^{\frac{\alpha n^2\pi^2 t}{a^2}},$$

which, after accounting for the initial condition, leads to

$$C_n(t) = \alpha \int_0^t Z_n(\tau) e^{\frac{\alpha n^2 \pi^2 \tau}{a^2}}\, d\tau. \tag{2.124}$$

Substituting for $C_n(t)$ into Eqn. (2.123a), we get

$$\frac{kT(r,t)}{\rho} = \frac{2\alpha}{a^2\zeta}\left\{\sum_{n=1}^{\infty} \sin(n\pi\zeta) \int_0^t \left[\int_0^a \hat{r} Q(\hat{r}, t-\tau) \sin\left(\frac{n\pi\hat{r}}{a}\right) d\hat{r}\right] e^{-\frac{\alpha n^2\pi^2\tau}{a^2}}\, d\tau\right\}. \tag{2.125}$$

For the case where Q is a function of t alone, we get

$$T = \sum_{n=1}^{\infty} \gamma_n \int_0^t Q(t-\tau) e^{-\frac{\alpha(n\pi)^2\tau}{a^2}}\, d\tau, \qquad (Q = Q(t)), \tag{2.126}$$

where

$$\gamma_n = \frac{2(-1)^{n+1}\rho\alpha}{k\zeta n\pi} \sin(n\pi\zeta),$$

while for the case where Q is a function of r alone, Eqn. (2.125) reduces to (compare against Eqn. (17), Section (9.8) of [2])

$$\frac{kT(r,t)}{\rho a^2} = \frac{2}{ar} \sum_{m=1}^{\infty} \frac{\left[1 - e^{-\alpha(m\pi)^2 t/a^2}\right] \sin(m\pi\zeta)}{(m\pi)^2} \int_0^a \hat{r} Q(\hat{r}) \sin\left(\frac{m\pi\hat{r}}{a}\right) d\hat{r}. \tag{2.127}$$

For $Q = Q_0$, the solution obtained using Eqn. (2.127) is given by

$$\frac{kT}{\rho Q_0 a^2} = \frac{1-\zeta^2}{6} + \frac{2}{\zeta} \sum_{n=1}^{\infty} \frac{(-1)^n \sin(n\pi\zeta) e^{-\alpha(n\pi)^2 t/a^2}}{(n\pi)^3}, \tag{2.128}$$

which agrees with the solution given by Eqn. (6), Section (9.8) of [2]. In the limit as $t \to \infty$, we recover the steady-state solution given by

$$\frac{kT}{\rho a^2 Q_0} = \frac{1-\zeta^2}{6}, \tag{2.129}$$

which agrees with Eqn. (2.110), and with Eqn. (12), Section (9.2) of [2].

For $Q(r) = Q_0(1-\zeta)$, we get

$$\frac{kT}{\rho a^2 Q_0} = \frac{(1-\zeta)(1+\zeta-\zeta^2)}{12} - \frac{8}{\zeta} \sum_{m=1}^{\infty} \frac{\sin[(2m-1)\pi\zeta] e^{-\alpha[(2m-1)\pi]^2 t/a^2}}{[(2m-1)\pi]^5}, \tag{2.130}$$

which agrees with the solution given by Eqn. (8), Section (9.8) of [2], while for $Q = Q_0(\zeta^2 - 1)$, we get

$$\frac{kT}{\rho a^2 Q_0} = \frac{(1-\zeta^2)(7-3\zeta^2)}{60} - \frac{12}{\zeta}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}\sin(m\pi\zeta)e^{-\alpha(m\pi)^2 t/a^2}}{(m\pi)^5}, \tag{2.131}$$

which agrees with the solution given by Eqn. (10), Section (9.8) of [2].

At the boundary, if in place of a prescribed temperature, we have a prescribed flux $q_a(t)$, then by using $-k\left(\partial\bar{T}/\partial r\right)_{r=a} = \bar{q}_a$, we now get

$$\bar{T} = \frac{a^2\bar{q}_a \sin qr}{kr\,(\sin qa - qa\cos qa)}, \tag{2.132}$$

The inversion of Eqn. (2.132) can be carried out similar to that of Eqn. (2.46). Let λ_n, $n = 1, 2, \ldots, \infty$, be the positive roots of $\tan x = x$. Assuming $q_a(t)$ to be arbitrary, we have

$$\begin{aligned}\bar{T} &= \frac{a^2\bar{q}_a \sin qr}{kr\,(\sin qa - qa\cos qa)}\\ &= \frac{a^2\bar{q}_a}{kr}\left[\frac{b}{s} + \sum_{n=1}^{\infty}\frac{\gamma_n}{s+\frac{\lambda_n^2\alpha}{a^2}}\right].\end{aligned}$$

Using Eqn. (1.100), we get

$$\begin{aligned}b &= -\frac{3\alpha r}{a^3},\\ \gamma_n &= -\frac{2\alpha\sin(\lambda_n\zeta)}{a^2\sin\lambda_n}.\end{aligned}$$

Thus,

$$T = -\frac{\alpha}{kr}\left[3\zeta\int_0^t q_a(\tau)\,d\tau + 2\sum_{n=1}^{\infty}\frac{\sin(\lambda_n\zeta)}{\sin\lambda_n}\int_0^t e^{-\alpha\lambda_n^2\tau/a^2}q_a(t-\tau)\,d\tau\right]. \tag{2.133}$$

If $Q = f(r) = 0$, $q_a(t) = q_0$, where q_0 is a constant, then, for $t \geq 0$, we get on using Eqn. (2.133)

$$\begin{aligned}\frac{kT}{q_0 a} &= -\left[\frac{3\alpha t}{a^2} + \frac{5\zeta^2-3}{10} - \frac{2}{\zeta}\sum_{n=1}^{\infty}\frac{\sin(\lambda_n\zeta)}{\lambda_n^2\sin\lambda_n}e^{-\frac{\alpha\lambda_n^2 t}{a^2}}\right], \qquad (r>0), && \text{(2.134a)}\\ &= -\left[\frac{3\alpha t}{a^2} - \frac{3}{10} - 2\sum_{n=1}^{\infty}\frac{1}{\lambda_n\sin\lambda_n}e^{-\frac{\alpha\lambda_n^2 t}{a^2}}\right], \qquad (r=0), && \text{(2.134b)}\end{aligned}$$

$$\frac{q}{q_0} = \zeta + \frac{2}{\zeta}\sum_{n=1}^{\infty} \frac{1}{\lambda_n^2 \sin\lambda_n}\left[\frac{1}{\zeta}\sin(\lambda_n\zeta) - \lambda_n\cos(\lambda_n\zeta)\right] e^{-\frac{\alpha\lambda_n^2 t}{a^2}}, \qquad (r > 0), \tag{2.134c}$$

$$= 0, \qquad (r = 0), \tag{2.134d}$$

which, after noting that $q_0 = -F_0$, is in agreement with Eqn. (1), Section (9.7) of [2].

If $Q = f(r) = 0$, and $q_a = q_0 e^{-\omega t}$, then the transient solution is given by substituting $q_a(t-\tau) = q_0 e^{-\omega(t-\tau)}$ into Eqn. (2.133), and with $\gamma := \sqrt{\omega a^2/\alpha}$, is given by

$$\frac{kT}{aq_0} = -\frac{3}{\gamma^2} + \frac{\sin(\gamma\zeta)e^{-\omega t}}{\zeta\,(\sin\gamma - \gamma\cos\gamma)} + \frac{2}{\zeta}\sum_{n=1}^{\infty}\frac{\sin(\lambda_n\zeta)e^{-\lambda_n^2\alpha t/a^2}}{\sin\lambda_n\,(\lambda_n^2 - \gamma^2)}. \tag{2.135}$$

If $Q = f(r) = 0$, and $q_a = q_0 \sin\omega t$, then the transient solution is given by substituting $q_a(t-\tau) = q_0 \sin\omega(t-\tau)$ into Eqn. (2.133), and with $\gamma := \sqrt{\omega a^2/(2\alpha)}$, is given by

$$\frac{kT}{aq_0} = C_0 + \frac{N_1}{D}\cos\omega t + \frac{N_2}{D}\sin\omega t - \sum_{n=1}^{\infty}\frac{4\gamma^2\sin(\lambda_n\zeta)}{\zeta\sin\lambda_n\,(\lambda_n^4 + 4\gamma^4)}e^{-\frac{\alpha\lambda_n^2 t}{a^2}}, \tag{2.136}$$

while for $q_a = q_0\cos\omega t$, we have

$$\frac{kT}{aq_0} = -\frac{N_1}{D}\sin\omega t + \frac{N_2}{D}\cos\omega t + \sum_{n=1}^{\infty}\frac{2\lambda_n^2\sin(\lambda_n\zeta)}{\zeta\sin\lambda_n\,(\lambda_n^4 + 4\gamma^4)}e^{-\frac{\alpha\lambda_n^2 t}{a^2}}, \tag{2.137}$$

where $C_0 = -3/(2\gamma^2)$, and

$$\begin{aligned}
N_1 &= 2\Big\{\cosh\gamma\zeta\sin\gamma\zeta\,[(\cos\gamma + \gamma\sin\gamma)\sinh\gamma - \gamma\cos\gamma\cosh\gamma] \\
&\qquad + \sinh\gamma\zeta\cos\gamma\zeta\,[\gamma\sin\gamma\sinh\gamma - (\sin\gamma - \gamma\cos\gamma)\cosh\gamma]\Big\}, \\
N_2 &= -2\Big\{\cosh\gamma\zeta\sin\gamma\zeta\,[(\gamma\cos\gamma - \sin\gamma)\cosh\gamma + \gamma\sin\gamma\sinh\gamma] \\
&\qquad + \sinh\gamma\zeta\cos\gamma\zeta\,[\gamma\cos\gamma\cosh\gamma - (\cos\gamma + \gamma\sin\gamma)\sinh\gamma]\Big\}, \\
D &= \zeta\left[(2\gamma^2 - 1)\cos 2\gamma + (1 + 2\gamma^2)\cosh 2\gamma - 2\gamma(\sinh 2\gamma + \sin 2\gamma)\right].
\end{aligned}$$

The 'periodic steady-state' solution is obtained by ignoring the last term in Eqns. (2.136) and (2.137). Note that although the average of the applied flux over a time-period is zero, due to the C_0 term, the average temperature over a time-period in this periodic solution is not zero!

Let λ_n, $n = 1, 2, \ldots, \infty$, denote the positive roots of $\tan x = x$. The evolution of the initial temperature field is given by

$$T = k_0 + \frac{1}{\zeta}\sum_{n=1}^{\infty} C_n \sin(\lambda_n\zeta)e^{-\frac{\alpha\lambda_n^2 t}{a^2}},$$

where, on imposing the initial conditions and using Eqns. (1.18) with $k = 0$, we have

$$k_0 = \frac{3}{a^3}\int_0^a \hat{r}^2 f(\hat{r})\,d\hat{r},$$
$$C_n = \frac{2}{a^2 \sin^2\lambda_n}\int_0^a \hat{r} f(\hat{r}) \sin\frac{\lambda_n \hat{r}}{a}\,d\hat{r}.$$

As an example, if $f(r) = C_0$, where C_0 is a constant, then we get $T = C_0$. As another example, if $f(r) = C_0\zeta^2$, then we get

$$\frac{T}{C_0} = \frac{3}{5} + \frac{4}{\zeta}\sum_{n=1}^{\infty} \frac{\sin(\lambda_n\zeta)}{\lambda_n^2 \sin\lambda_n} e^{-\alpha\lambda_n^2 t/a^2}. \tag{2.138}$$

As we can see from the above solution and consistent with Eqn. (1.113), the steady-state temperature is the average of the initial temperature distribution $f(r)$.

Similarly, if $f(r) = q_a = 0$, but $Q(r,t)$ is nonzero, then the form of the solution is given by

$$T = C_0(t) + \frac{1}{\zeta}\sum_{n=1}^{\infty} C_n(t)\sin(\lambda_n\zeta) e^{-\frac{\alpha\lambda_n^2 t}{a^2}}, \tag{2.139a}$$

$$\frac{\rho Q}{k} = Z_0(t) + \frac{1}{\zeta}\sum_{n=1}^{\infty} Z_n(t)\sin(\lambda_n\zeta), \tag{2.139b}$$

where

$$Z_0(t) = \frac{3}{a^3}\int_0^a \hat{r}^2 Q(\hat{r},t)\,d\hat{r},$$
$$Z_n(t) = \frac{2\rho}{ka^2 \sin^2\lambda_n}\int_0^a \hat{r} Q(\hat{r},t) \sin\frac{\lambda_n \hat{r}}{a}\,d\hat{r}.$$

Substituting Eqns. (2.139) into Eqn. (2.111), we get

$$C_0'(t) = \alpha Z_0(t),$$
$$C_n'(t) = \alpha Z_n(t) e^{\frac{\alpha\lambda_n^2 t}{a^2}},$$

which, after accounting for the initial condition, leads to

$$C_0(t) = \alpha\int_0^t Z_0(\tau)\,d\tau, \tag{2.140a}$$

$$C_n(t) = \alpha\int_0^t Z_n(\tau) e^{\frac{\alpha\lambda_n^2 \tau}{a^2}}\,d\tau. \tag{2.140b}$$

If $Q(r,t)$ is a function of time alone, then we get

$$\frac{kT}{\rho} = \alpha \int_0^t Q(\tau)\, d\tau, \qquad (Q = Q(t),\ f(r) = q_a = 0). \tag{2.141}$$

As an example, if $Q(r,t) = \zeta Q_0$, then we get

$$\frac{kT(r,t)}{\rho Q_0 a^2} = \frac{3\alpha t}{4a^2} + \frac{15\zeta^2 - 10\zeta^3 - 4}{120} - \frac{2}{\zeta}\sum_{m=1}^{\infty} \frac{\sin(\lambda_m \zeta)\tan^2(\lambda_m/2) e^{-\alpha\lambda_m^2 t/a^2}}{\lambda_m^4 \sin\lambda_m}. \tag{2.142}$$

As another example, if $Q(r,t) = Q_0(1 - 4\zeta/3)$, then we get

$$\frac{kT}{\rho Q_0 a^2} = \frac{4 + 5\zeta^2(2\zeta - 3)}{90} + \sum_{m=1}^{\infty} \frac{8\tan^2(\lambda_m/2)\sin(\lambda_m\zeta) e^{-\alpha\lambda_m^2 t/a^2}}{3\lambda_m^5 \zeta\cos\lambda_m}. \tag{2.143}$$

Chapter 3

Hyperbolic partial differential equations

We shall deal mainly with the acoustic wave equation. However, the equation for the transverse vibration of a membrane is very similar, and the solutions for the acoustic wave equation can be used to generate solutions for membrane vibration problems also. In the previous chapter, we first considered solution to steady-state problems, and then to transient problems. In this chapter, we first consider 'periodic steady-state' solutions before proceeding to the solution of transient problems.

The relation between the velocity and the pressure fields is

$$\rho \frac{\partial \boldsymbol{u}}{\partial t} = -\boldsymbol{\nabla} p_{\Delta}, \tag{3.1}$$

so that the acceleration $\partial \boldsymbol{u}/\partial t$ is analogous to the flux in heat conduction problems. For finding the periodic steady-state solutions, we assume $p_{\Delta} = \tilde{p}(\boldsymbol{x})e^{i\omega t}$ and $\boldsymbol{u} = \tilde{\boldsymbol{u}}(\boldsymbol{x})e^{i\omega t}$, and solve for the field $\tilde{p}(\boldsymbol{x})$, after which one can obtain $\tilde{\boldsymbol{u}}(\boldsymbol{x})$ using Eqn. (3.1) as

$$\tilde{\boldsymbol{u}} = -\frac{\boldsymbol{\nabla} \tilde{p}}{\rho i \omega}. \tag{3.2}$$

Since $\tilde{\boldsymbol{u}}(\boldsymbol{x})$ can be determined from the above equation once $\tilde{p}(\boldsymbol{x})$ is determined, we shall focus on finding $\tilde{p}(\boldsymbol{x})$. The physical solution p_{Δ} under cosine or sine loading is obtained as the real and imaginary parts of $\tilde{p}(\boldsymbol{x})e^{i\omega t}$, with a similar procedure for finding $\boldsymbol{u}$ from $\tilde{\boldsymbol{u}}(\boldsymbol{x})$. If a_0 denotes the acoustic wave speed, then $k := \omega/a_0$ is used to denote the wave number throughout this chapter. A rigid boundary implies that the normal velocity component $\boldsymbol{u} \cdot \boldsymbol{n}$ is zero (and hence $(\boldsymbol{\nabla} p_{\Delta}) \cdot \boldsymbol{n} = 0$) along such a surface.

3.1 Straight duct with specified accelerations or pressures at the two ends

Let the length of the duct be h, and let $\xi := x/h$.

The periodic steady-state solution for the case when the acceleration and pressure is specified at the ends $x = 0$ and at $x = h$ as $A(t) = Vi\omega e^{i\omega t}$ and $P(t) = P_0 e^{i\omega t}$ is obtained from Eqn. (1.87) by assuming the solution form to be $\tilde{p} = c_1 \sin kh\xi + c_2 \cos kh\xi$, and then determining the constants c_1 and c_2 using the boundary conditions as

$$\tilde{p} = \left[\frac{i\rho a_0 V \sin[kh(1-\xi)]}{\cos kh} + \frac{P_0 \cos(kh\xi)}{\cos kh} \right]. \tag{3.3}$$

Similarly, the periodic steady-state solution for the case when $A(t) = Vi\omega e^{i\omega t}$ at the end $x = 0$, with the end $x = h$ being rigid is

$$\frac{\tilde{p}}{\rho a_0 V} = -\frac{i \cos[kh(1-\xi)]}{\sin kh}. \tag{3.4}$$

If the pressure is specified as $P = P_0 e^{i\omega t}$ and $P = 0$ at the ends $x = 0$ and $x = h$, respectively, then the periodic steady state solution is

$$\frac{\tilde{p}}{P_0} = \frac{\sin[kh(1-\xi)]}{\sin kh}. \tag{3.5}$$

Now we turn to the solution of transient problems. First consider the case when the domain is unbounded, i.e., $h \to \infty$. We assume that the fluid is initially stationary, i.e., both u and p_Δ are assumed to be zero at $t = 0$, and that the normal acceleration a_x is specified as $A(t)$ at $x = 0$. Under the one-dimensional flow approximation being made, Eqn. (1.121) reduces to

$$\frac{\partial^2 p_\Delta}{\partial x^2} = \frac{1}{a_0^2} \frac{\partial^2 p_\Delta}{\partial t^2}. \tag{3.6}$$

Taking the Laplace transform of Eqn. (3.6), we get

$$\frac{\partial^2 \bar{p}_\Delta}{\partial x^2} - \frac{s^2 \bar{p}_\Delta}{a_0^2} = 0,$$

whose general solution is $c_1(s)e^{qx} + c_2(s)e^{-qx}$, where $q = s/a_0$. Since $p_\Delta \to 0$ as $x \to \infty$, we get $c_1 = 0$. From the boundary condition at $x = 0$, we get $c_2 = \rho \bar{A}(s)/q = \rho a_0 \bar{A}(s)/s$. Thus, we have

$$\frac{\bar{p}_\Delta}{\rho a_0} = \frac{\bar{A}(s) e^{-sx/a_0}}{s}, \tag{3.7}$$

Using the convolution theorem, we get

$$\frac{p_\Delta}{\rho a_0} = H(t_s) \int_0^{t_s} A\left(t_s - \tau\right)\, d\tau = H(t_s) \int_0^{t_s} A(\tau)\, d\tau, \tag{3.8}$$

where $H(.)$ denotes the Heaviside function, and $t_s := t - x/a_0$ is the 'shifted time'. For an applied constant acceleration A_0, we get $p_\Delta = 0$ for $t_s < 0$ and $p_\Delta/(\rho a_0) = A_0 t_s$ for $t_s \geq 0$. For $A(t) = V\omega \cos \omega t$, we get $p_\Delta = 0$ for $t_s < 0$ and $p_\Delta/(\rho a_0 V) = \sin(\omega t_s)$ for $t_s \geq 0$. Similarly, for $A(t) = V\omega \sin(\omega t)$, we get $p_\Delta = 0$ for $t_s < 0$ and $p_\Delta/(\rho a_0 V) = 1 - \cos(\omega t_s)$ for $t_s \geq 0$.

If the normal velocity is zero at $x = 0$, and the initial conditions are $p_\Delta(x, 0) = p_0(x)$ and $\dot{p}_\Delta(x, 0) = v_0(x)$, then we assume the solution form to be

$$p_\Delta = \int_0^\infty \cos \lambda x \left[A(\lambda) \cos(\lambda a_0 t) + B(\lambda) \sin(\lambda a_0 t)\right]\, d\lambda. \tag{3.9}$$

Imposing the initial conditions, we get

$$\begin{aligned} \int_0^\infty A(\lambda) \cos \lambda x\, d\lambda &= p_0(x), \\ \int_0^\infty \lambda B(\lambda) \cos \lambda x\, d\lambda &= \frac{v_0(x)}{a_0}, \end{aligned}$$

from which we get

$$\begin{aligned} A(\lambda) &= \frac{2}{\pi} \int_0^\infty p_0(\hat{x}) \cos(\lambda \hat{x})\, d\hat{x}, \\ B(\lambda) &= \frac{2}{\pi a_0 \lambda} \int_0^\infty v_0(\hat{x}) \cos(\lambda \hat{x})\, d\hat{x}. \end{aligned}$$

As an example, if $v_0 = 0$, and $p_0(x) = p_0$ for $x \leq a$, and zero elsewhere, then we get $B(\lambda) = 0$, and

$$\frac{A(\lambda)}{p_0} = \frac{2 \sin(\lambda a)}{\pi \lambda},$$

which on substituting into Eqn. (3.9), and carrying out the integration with respect to λ yields

$$\frac{p_\Delta}{p_0} = \begin{cases} \dfrac{2 + \operatorname{sgn}(a - x - a_0 t) + \operatorname{sgn}(a + x - a_0 t)}{4}, & 0 < x \leq a, \\ \dfrac{\operatorname{sgn}(a - x + a_0 t) + \operatorname{sgn}(a + x - a_0 t)}{4}, & 0 < a \leq x. \end{cases} \tag{3.10}$$

If $\tilde{p}|_{t=0} = \dot{\tilde{p}}|_{t=0} = 0$, if the normal velocity is zero at $x = 0$, and if we consider the non-homogeneous wave equation given by

$$\frac{\partial^2 \tilde{p}}{\partial x^2} + G(x,t) = \frac{1}{a_0^2} \frac{\partial^2 \tilde{p}}{\partial t^2}, \tag{3.11}$$

then, analogous to the method of variation of parameters, we assume the solution form to be the following modified form of Eqn. (3.9):

$$\tilde{p}(x,t) = \int_0^\infty A(\lambda,t) \cos \lambda x \, d\lambda, \tag{3.12a}$$

$$G(x,t) = \int_0^\infty R(\lambda,t) \cos \lambda x \, d\lambda. \tag{3.12b}$$

Note that the assumed form for $\tilde{p}$ automatically satisfies the boundary conditions. Substituting Eqns. (3.12) into Eqn. (3.11), and with a prime denoting a derivative with respect to t, we get

$$A''(\lambda,t) + (\lambda a_0)^2 A(\lambda,t) = a_0^2 R(\lambda,t).$$

The above equation can be solved using the method of variation of parameters, or more conveniently, using a Laplace transform. Thus, taking the Laplace transform of the above equation, and in view of the initial conditions $A(\lambda,0) = \dot{A}(\lambda,0) = 0$, we get

$$\bar{A}(\lambda,s) = \frac{a_0^2 \bar{R}(\lambda,s)}{s^2 + (\lambda a_0)^2},$$

which on using the Laplace transform for a sine function and the convolution theorem yields

$$A(\lambda,t) = \frac{a_0}{\lambda} \int_0^t \sin[a_0 \lambda (t-\tau)] R(\lambda,\tau) \, d\tau. \tag{3.13}$$

From Eqn. (3.12b), we get

$$R(\lambda,t) = \frac{2}{\pi} \int_0^\infty G(\hat{x},t) \cos(\lambda \hat{x}) \, d\hat{x},$$

which when substituted into Eqn. (3.13) yields $A(\lambda,t)$, and hence $\tilde{p}(x,t)$.

If in place of the normal acceleration, the pressure itself is specified at the end $x = 0$, i.e., $p_\Delta(0,t) = P(t)$, then in place of Eqns. (3.7), we get

$$\bar{p}_\Delta = \bar{P}(s) e^{-sx/a_0}.$$

Inverting the above transform, we get

$$p_\Delta(x,t) = P(t_s) H(t_s). \tag{3.14}$$

Thus, $p_\Delta = 0$ for $t_s > 0$ and $p(x,t) = P(t_s)$ for $t_s > 0$. If $P(0) \neq 0$, then there is a jump discontinuity in the pressure $p_\Delta(x,t)$ at $t_s = 0$, which is not permissible since then $\partial p_\Delta/\partial x$ would be discontinuous and $\partial^2 p_\Delta/\partial x^2$ would be infinite (one can also argue physically that $p(x,t)$ is a continuous function of x and t). Thus, the prescribed pressure must be such that $P(0) = 0$; for example, $P(t) = P_0\,(1 - \cos\omega t)$ or $P(t) = P_0\,(1 - e^{-\omega t})$ are permissible, but $P(t) = P_0 \cos\omega t$ is not.

If the pressure is zero at $x = 0$, and the initial conditions are $p_\Delta(x,0) = p_0(x)$ and $\dot{p}_\Delta(x,0) = v_0(x)$, then in place of Eqn. (3.9), we now have

$$p_\Delta = \int_0^\infty \sin\lambda x \left[A(\lambda)\cos(\lambda a_0 t) + B(\lambda)\sin(\lambda a_0 t)\right]. \tag{3.15}$$

Imposing the initial conditions, we get

$$\int_0^\infty A(\lambda)\sin\lambda x\, d\lambda = p_0(x),$$
$$\int_0^\infty \lambda B(\lambda)\sin\lambda x\, d\lambda = \frac{v_0(x)}{a_0},$$

which on inverting yields

$$A(\lambda) = \frac{2}{\pi}\int_0^\infty p_0(\hat{x})\sin(\lambda\hat{x})\, d\hat{x},$$
$$B(\lambda) = \frac{2}{\pi a_0 \lambda}\int_0^\infty v_0(\hat{x})\sin(\lambda\hat{x})\, d\hat{x}.$$

As an example, if $v_0(x) = 0$ and $p_0(x) = p_0$ for $0 \le x \le a$ and zero elsewhere, then we get $B(\lambda) = 0$, and

$$\frac{A(\lambda)}{p_0} = \frac{2(1 - \cos\lambda a)}{\pi\lambda},$$

which on substituting into Eqns. (3.15), and carrying out the integrations with respect to λ yields

$$\frac{p_\Delta}{p_0} = \begin{cases} \dfrac{2 + \operatorname{sgn}(a - x - a_0 t) - \operatorname{sgn}(a + x - a_0 t) - 2\operatorname{sgn}(a_0 t - x)}{4}, & 0 < x \le a, \\[2ex] \dfrac{\operatorname{sgn}(a - x + a_0 t) - \operatorname{sgn}(a + x - a_0 t) - 2\operatorname{sgn}(a_0 t - x)}{4}, & 0 < a \le x. \end{cases} \tag{3.16}$$

If $\tilde{p}\big|_{t=0} = \dot{\tilde{p}}\big|_{t=0} = \tilde{p}\big|_{x=0} = 0$, and if we consider the non-homogeneous wave equation given by Eqn. (3.11), then we assume the solution form to be

$$\tilde{p}(x,t) = \int_0^\infty A(\lambda,t)\sin\lambda x\, d\lambda, \tag{3.17a}$$

$$G(x,t)=\int_0^\infty R(\lambda,t)\sin\lambda x\,d\lambda. \tag{3.17b}$$

Note that the assumed form for $\tilde{p}$ automatically satisfies the boundary conditions. Substituting Eqns. (3.17) into Eqn. (3.11), and with a prime denoting a derivative with respect to t, we get

$$A''(\lambda,t)+(\lambda a_0)^2A(\lambda,t)=a_0^2R(\lambda,t),$$

whose solution obtained using the Laplace transform method and the initial conditions (as in the derivation of Eqn. (3.13)) is

$$A(\lambda,t)=\frac{a_0}{\lambda}\int_0^t \sin[a_0\lambda(t-\tau)]R(\lambda,\tau)\,d\tau. \tag{3.18}$$

From Eqn. (3.17b), we get

$$R(\lambda,t)=\frac{2}{\pi}\int_0^\infty G(\hat{x},t)\sin(\lambda\hat{x})\,d\hat{x},$$

which when substituted into Eqn. (3.18) yields $A(\lambda,t)$, and hence $\tilde{p}(x,t)$.

Now consider the case when h is finite. First consider the case where the acceleration $A(t)$ is specified at $x=0$, and the pressure $P(t)$ is specified at $x=h$. The general solution for $\bar{p}_\Delta$ is $c_1\sin qx+c_2\cos qx$, where $q^2=-s^2/a_0^2$. Using the boundary conditions at the two ends, we get

$$\bar{p}_\Delta=\frac{\rho\bar{A}(s)\sin q(h-x)}{q\cos qh}+\frac{\bar{P}\cos qx}{\cos qh}.$$

Since the roots of $\cos qh$ are given by $qh=(2n-1)\pi/2$, $n=1,2,\ldots,\infty$, we can write

$$\frac{\sin q(h-x)}{q\cos qh}=\sum_{n=1}^\infty \frac{c_n^{(1)}}{s+\lambda_n a_0 i}+\frac{c_n^{(2)}}{s-\lambda_n a_0 i},$$

where $\lambda_n:=(2n-1)\pi/(2h)$, and where $c_n^{(1)}$ and $c_n^{(2)}$ are found using the Heaviside formula, and the fact that $dq/ds=-s/(a_0^2q)$ as

$$\begin{aligned} c_n^{(1)}&=-\left.\frac{\sin[\lambda_n(h-x)]}{qh\sin(\lambda_n h)\frac{dq}{ds}}\right|_{q=\lambda_n}=\frac{2ia_0\cos\lambda_n x}{(2n-1)\pi},\\ c_n^{(2)}&=-c_n^{(1)}. \end{aligned}$$

On using the convolution theorem, we get

$$\begin{aligned} p_\Delta=\rho a_0&\sum_{n=1}^\infty \frac{4\cos(\lambda_n x)}{(2n-1)\pi}\int_0^t \sin(\lambda_n a_0\tau)A(t-\tau)\,d\tau\\ &-\frac{2a_0}{h}\sum_{n=1}^\infty(-1)^n\cos(\lambda_n x)\int_0^t \sin(\lambda_n a_0\tau)P(t-\tau)\,d\tau. \end{aligned} \tag{3.19}$$

For the case of a constant applied acceleration and pressure, i.e., when $A(t) = A_0$ and $P(t) = P_0$, we get with $\xi := x/h$,

$$p_\Delta = \rho h A_0 \left[1 - \xi - \sum_{n=1}^{\infty} \frac{8\cos(\lambda_n x)\cos(\lambda_n a_0 t)}{(2n-1)^2\pi^2}\right] + P_0 \left[1 + \sum_{n=1}^{\infty} \frac{4(-1)^n \cos(\lambda_n x)\cos(\lambda_n a_0 t)}{(2n-1)\pi}\right], \tag{3.20}$$

where we have used

$$\sum_{n=1}^{\infty} \frac{8\cos(\lambda_n x)}{(2n-1)^2\pi^2} = 1 - \xi. \tag{3.21}$$

Note that because the acceleration is suddenly applied, there is a mismatch between the initial and boundary conditions on the acceleration at $x = 0$ (in the sense that $A(0) \neq 0$). Consistent with this mismatch, term-by-term differentiation of Eqn. (3.21) is valid everywhere except at $x = 0$[1], i.e.,

$$\sum_{n=1}^{\infty} \frac{4\sin\lambda_n x}{(2n-1)\pi} = \begin{cases} 1, & x \in (0, h] \\ 0, & x = 0. \end{cases}$$

The solutions given by Eqns. (3.20) is valid for all $x \in [0, h]$. However, while evaluating $\partial p/\partial x$ by term-by-term differentiation of this solution, we see by virtue of the above equation that at $t = 0$, $\partial p/\partial x = 0$ for all $x \in (0, h]$, while it is equal to A_0 at $x = 0$. This 'discontinuity' is due to the aforementioned mismatch between the initial and boundary conditions, and transmits downstream with increasing t; nevertheless, apart from this moving discontinuity point (where $\partial p/\partial x$ is indeterminate), term-by-term differentiation yields the correct solution over the entire domain! However, as discussed after Eqn. (3.14), although a discontinuity is allowed in the applied acceleration, it is not allowed in the applied pressure at $x = h$, i.e., we need $P(0) = 0$. Thus, the above solution for $P(t) = P_0$ should not be used in isolation, but in superposition with other prescribed pressures such that finally $P(0) = 0$; for example $P(t) = P_0(1 - \cos\omega t)$ or $P(t) = P_0(1 - e^{-\alpha t})$ etc.

When $A(t) = V\alpha e^{-\alpha t}$ and $P(t) = P_0 e^{-\alpha t}$, where α is a positive constant, we get (with $k \equiv \alpha/a_0$)

$$p_\Delta = \rho a_0 V \left\{ \frac{\sinh[kh(1-\xi)]}{\cosh kh} e^{-\alpha t} + \sum_{n=1}^{\infty} \frac{\cos(\lambda_n x)}{\left[\left(\frac{\lambda_n}{k}\right)^2 + 1\right]} \left[\frac{4\sin\lambda_n a_0 t}{(2n-1)\pi} - \frac{2\cos\lambda_n a_0 t}{kh}\right]\right\}$$

[1]Similar constraints on term-by-term differentiation arise whenever the specified acceleration $A(t)$ is such that $A(0) \neq 0$.

$$+ P_0 \left[\frac{\cosh kh\xi}{\cosh kh} e^{-\alpha t} + \frac{2}{kh} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\lambda_n x) \left(\frac{\lambda_n}{k} \cos(\lambda_n a_0 t) - \sin(\lambda_n a_0 t) \right)}{\left(\frac{\lambda_n}{k} \right)^2 + 1} \right].$$

For the case $A(t) = V\omega \cos \omega t$ and $P(t) = P_0 \cos \omega t$, we get (with $k \equiv \omega / a_0$)

$$p_\Delta = \rho a_0 V \left[\frac{\sin[kh(1-\xi)] \cos \omega t}{\cos kh} - \frac{2}{kh} \sum_{n=1}^{\infty} \frac{\cos(\lambda_n x) \cos(\lambda_n a_0 t)}{\left(\frac{\lambda_n}{k} \right)^2 - 1} \right]$$
$$+ P_0 \left[\frac{\cos(kh\xi) \cos \omega t}{\cos kh} + \frac{2}{kh} \sum_{n=1}^{\infty} \frac{(-1)^n \left(\frac{\lambda_n}{k} \right) \cos(\lambda_n x) \cos(\lambda_n a_0 t)}{\left(\frac{\lambda_n}{k} \right)^2 - 1} \right], \tag{3.22a}$$

while for the case $A(t) = V\omega \sin \omega t$ and $P(t) = P_0 \sin \omega t$, we get

$$p_\Delta = \rho a_0 V \left[\frac{\sin[kh(1-\xi)] \sin \omega t}{\cos kh} - \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi} \frac{\cos(\lambda_n x) \sin \lambda_n a_0 t}{\left(\frac{\lambda_n}{k} \right)^2 - 1} \right]$$
$$+ P_0 \left[\frac{\cos(kh\xi) \sin \omega t}{\cos kh} + \frac{2}{kh} \sum_{n=1}^{\infty} \frac{(-1)^n \cos(\lambda_n x) \sin(\lambda_n a_0 t)}{\left(\frac{\lambda_n}{k} \right)^2 - 1} \right]. \tag{3.23a}$$

If $P(t) = 0$, if the surface $x = 0$ is rigid, and if the initial conditions are given by $p_\Delta(x, 0) = p_0(x)$ and $\dot{p}_\Delta(x, 0) = v_0(x)$, then with $\lambda_n := (2n-1)\pi/2$, we assume the solution form to be given by

$$p_\Delta = \sum_{n=1}^{\infty} \cos(\lambda_n \xi) \left(A_n \cos \frac{\lambda_n a_0 t}{h} + B_n \sin \frac{\lambda_n a_0 t}{h} \right). \tag{3.24}$$

Imposing the initial conditions, we get

$$\sum_{n=1}^{\infty} A_n \cos(\lambda_n \xi) = p_0(\xi), \tag{3.25a}$$

$$\sum_{n=1}^{\infty} \lambda_n B_n \cos(\lambda_n \xi) = \frac{h v_0(\xi)}{a_0}, \tag{3.25b}$$

which, on using the orthogonality of the sine and cosine functions, yields

$$A_n = 2 \int_0^1 p_0(\hat{\xi}) \cos(\lambda_n \hat{\xi}) \, d\hat{\xi},$$
$$B_n = \frac{h}{a_0 \lambda_n} \int_0^1 v_0(\hat{\xi}) \cos(\lambda_n \hat{\xi}) \, d\hat{\xi}.$$

As an example, if $v_0(x) = 0$, and $p_0(x) = p_0$ for $0 \le x \le h/2$ and zero elsewhere, then we get $B_n = 0$, and

$$\frac{A_n}{p_0} = \frac{4}{\pi(1-2n)} \cos \frac{\pi(2n+1)}{4},$$

which yields the solution

$$\frac{p}{p_0} = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{1}{(1-2n)} \cos \frac{\pi(2n+1)}{4} \cos(\lambda_n \xi) \cos \frac{\lambda_n a_0 t}{h}. \tag{3.26}$$

If $\tilde{p}|_{t=0} = \dot{\tilde{p}}\big|_{t=0} = P(t) = 0$, if the surface $x = 0$ is rigid, and if we consider the non-homogeneous wave equation given by Eqn. (3.11), then we assume the solution form to be

$$\tilde{p}(x,t) = \sum_{n=1}^{\infty} A_n(t) \cos \lambda_n \xi, \tag{3.27a}$$

$$G(x,t) = \sum_{n=1}^{\infty} R_n(t) \cos \lambda_n \xi. \tag{3.27b}$$

Note that the assumed form for $\tilde{p}$ automatically satisfies the boundary conditions. Substituting Eqns. (3.27) into Eqn. (3.11), and with a prime denoting a derivative with respect to t, we get

$$A_n''(t) + \left(\frac{\lambda_n a_0}{h}\right)^2 A_n(t) = a_0^2 R_n(t),$$

whose solution obtained using the Laplace transform method and the initial conditions is

$$A_n(t) = \frac{a_0 h}{\lambda_n} \int_0^t \sin\left[\frac{a_0 \lambda_n (t-\tau)}{h}\right] R_n(\tau)\, d\tau. \tag{3.28}$$

From Eqn. (3.27b), we get

$$R_n(t) = 2 \int_0^1 G(\hat{\xi}, t) \cos(\lambda_n \hat{\xi})\, d\hat{\xi},$$

which when substituted into Eqn. (3.28) yields $A_n(t)$, and hence $\tilde{p}(x,t)$.

If at the right end, instead of a specified pressure, the acceleration is prescribed to be zero, with the acceleration still prescribed to be $A(t)$ at the left end, then we have

$$\bar{p}_\Delta = -\frac{\rho \bar{A}(s) \cos q(h-x)}{q \sin qh}.$$

On inverting the above transforms, we get

$$\frac{p_\Delta}{\rho a_0} = \frac{a_0}{h}\int_0^t \tau A(t-\tau)\,d\tau + 2\sum_{n=1}^{\infty}\frac{\cos(\lambda_n x)}{n\pi}\int_0^t \sin(\lambda_n a_0 \tau)A(t-\tau)\,d\tau, \tag{3.29}$$

where $\lambda_n := n\pi/h$.

For the case of an applied step acceleration, i.e., when $A(t) = A_0$, we get

$$\frac{p_\Delta}{\rho h A_0} = \frac{1}{3} - \xi + \frac{(h\xi)^2 + a_0^2 t^2}{2h^2} - \sum_{n=1}^{\infty}\frac{2\cos(\lambda_n x)\cos(\lambda_n a_0 t)}{(n\pi)^2}.$$

When $A(t) = V\alpha e^{-\alpha t}$, where α is a positive constant, we get (with $k \equiv \alpha/a_0$)

$$\begin{aligned}\frac{p_\Delta}{\rho a_0 V} &= \frac{\alpha t - 1}{kh} + \frac{\cosh[kh(1-\xi)]e^{-\alpha t}}{\sinh kh} \\ &\quad + \sum_{n=1}^{\infty}\frac{2\cos(\lambda_n x)}{\left[\left(\frac{\lambda_n}{k}\right)^2 + 1\right]}\left[\frac{1}{n\pi}\sin\lambda_n a_0 t - \frac{1}{kh}\cos\lambda_n a_0 t\right].\end{aligned}$$

For the case $A(t) = V\omega\cos\omega t$, we get (with $k \equiv \omega/a_0$)

$$\frac{p_\Delta}{\rho a_0 V} = \frac{1}{kh} - \frac{\cos[kh(1-\xi)]\cos\omega t}{\sin kh} - \frac{1}{kh}\sum_{n=1}^{\infty}\frac{2\cos(\lambda_n x)\cos(\lambda_n a_0 t)}{\left(\frac{n\pi}{kh}\right)^2 - 1},$$

while for the case $A(t) = V\omega\sin\omega t$, we get

$$\frac{p_\Delta}{\rho a_0 V} = \frac{\omega t}{kh} - \frac{\cos[kh(1-\xi)]\sin\omega t}{\sin kh} - \sum_{n=1}^{\infty}\frac{2}{n\pi}\frac{\cos(\lambda_n x)\sin(\lambda_n a_0 t)}{\left(\frac{n\pi}{kh}\right)^2 - 1}.$$

If both ends of the duct are rigid, and the initial conditions are $p_\Delta(x,0) = p_0(x)$ and $\dot{p}_\Delta(x,0) = v_0(x)$, then with $\lambda_n := n\pi$, we now have

$$p_\Delta = A_0 + \sum_{n=1}^{\infty}\cos(\lambda_n\xi)\left(A_n\cos\frac{\lambda_n a_0 t}{h} + B_n\sin\frac{\lambda_n a_0 t}{h}\right), \tag{3.30}$$

Imposing the initial conditions, we get

$$\begin{aligned}A_0 + \sum_{n=1}^{\infty}A_n\cos(\lambda_n\xi) &= p_0(\xi), \\ \sum_{n=1}^{\infty}\lambda_n B_n\cos(\lambda_n\xi) &= \frac{h v_0(\xi)}{a_0},\end{aligned}$$

which, on using the orthogonality of the sine and cosine functions, yields

$$\begin{aligned} A_0 &= \int_0^1 p_0(\hat{\xi})\, d\hat{\xi}, \\ A_n &= 2\int_0^1 p_0(\hat{\xi}) \cos(\lambda_n \hat{\xi})\, d\hat{\xi}, \\ B_n &= \frac{2h}{a_0 \lambda_n} \int_0^1 v_0(\hat{\xi}) \cos(\lambda_n \hat{\xi})\, d\hat{\xi}. \end{aligned}$$

As an example, if $v_0(x) = 0$, and $p_0(x) = p_0$ for $0 \leq x \leq h/2$ and zero elsewhere, then we get $B_n = 0$, $A_0 = p_0/2$, and

$$\frac{A_n}{p_0} = \frac{2}{n\pi} \sin \frac{n\pi}{2},$$

which yields the solution

$$\frac{p}{p_0} = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(n\pi/2)}{n} \cos\left(\frac{\lambda_n x}{h}\right) \cos \frac{\lambda_n a_0 t}{h}.$$

If $\tilde{p}|_{t=0} = \dot{\tilde{p}}\big|_{t=0} = 0$, if both ends of the duct $x = 0$ and $x = L$ are rigid, and if we consider the non-homogeneous wave equation given by Eqn. (3.11), then we assume the solution form to be

$$\tilde{p}(x,t) = A_0(t) + \sum_{n=1}^{\infty} A_n(t) \cos \lambda_n \xi, \tag{3.31a}$$

$$G(x,t) = R_0(t) + \sum_{n=1}^{\infty} R_n(t) \cos \lambda_n \xi. \tag{3.31b}$$

Note that the assumed form for $\tilde{p}$ automatically satisfies the boundary conditions. Substituting Eqns. (3.31) into Eqn. (3.11), and with a prime denoting a derivative with respect to t, we get

$$\begin{aligned} A_0''(t) &= a_0^2 R_0(t), \\ A_n''(t) + \left(\frac{\lambda_n a_0}{h}\right)^2 A_n(t) &= a_0^2 R_n(t), \end{aligned}$$

whose solution obtained using the Laplace transform method and the initial conditions is

$$\begin{aligned} A_0(t) &= a_0^2 \int_0^t \tau R_0(t-\tau)\, d\tau, \\ A_n(t) &= \frac{a_0 h}{\lambda_n} \int_0^t \sin\left[\frac{a_0 \lambda_n (t-\tau)}{h}\right] R_n(\tau)\, d\tau. \end{aligned} \tag{3.32}$$

From Eqn. (3.31b), we get

$$R_0(t) = \int_0^1 G(\hat{\xi}, t)\, d\hat{\xi},$$
$$R_n(t) = 2\int_0^1 G(\hat{\xi}, t)\cos(\lambda_n \hat{\xi})\, d\hat{\xi},$$

which when substituted into Eqns. (3.32) yield $A_0(t)$ and $A_n(t)$, and hence $\tilde{p}(x, t)$.

3.2 Three-dimensional half-space

If the surface $z = 0$ is rigid, and if the initial conditions are $p_\Delta(r, z, 0) = p_0(r, z)$ and $\dot{p}_\Delta(r, z, 0) = v_0(r, z)$, then the form of the solution is

$$p_\Delta = \int_0^\infty \int_0^\infty J_0(\lambda r)\cos(\gamma z)\Big[A(\lambda,\gamma)\cos(a_0 t\sqrt{\lambda^2+\gamma^2}) + B(\lambda,\gamma)\sin(a_0 t\sqrt{\lambda^2+\gamma^2})\Big]\, d\lambda\, d\gamma. \tag{3.33}$$

Imposing the initial conditions, we get

$$\int_0^\infty \int_0^\infty A(\lambda,\gamma) J_0(\lambda r)\cos(\gamma z)\, d\lambda\, d\gamma = p_0(r, z),$$
$$\int_0^\infty \int_0^\infty \sqrt{\lambda^2+\gamma^2} B(\lambda,\gamma) J_0(\lambda r)\cos(\gamma z)\, d\lambda\, d\gamma = \frac{v_0(r, z)}{a_0}.$$

which on inverting yields

$$\begin{aligned} A(\lambda,\gamma) &= \frac{2\lambda}{\pi}\int_0^\infty \int_0^\infty \hat{r} p_0(\hat{r}, \hat{z}) J_0(\lambda\hat{r})\cos(\gamma\hat{z})\, d\hat{r}\, d\hat{z}, \\ B(\lambda,\gamma) &= \frac{2\lambda}{\pi a_0\sqrt{\lambda^2+\gamma^2}}\int_0^\infty \int_0^\infty \hat{r} v_0(\hat{r}, \hat{z}) J_0(\lambda\hat{r})\cos(\gamma\hat{z})\, d\hat{r}\, d\hat{z}. \end{aligned} \tag{3.34}$$

If $p_\Delta|_{z=0} = 0$, and if the initial conditions are $p_\Delta(r, z, 0) = p_0(r, z)$ and $\dot{p}_\Delta(r, z, 0) = v_0(r, z)$, then $\cos\gamma z$ and $\cos\gamma\hat{z}$ are, respectively, replaced by $\sin\gamma z$ and $\sin\gamma\hat{z}$ in Eqns. (3.33)–(3.34).

If $\tilde{p}|_{t=0} = \dot{\tilde{p}}|_{t=0} = 0$, if the surface $z = 0$ is rigid, and if we consider the non-homogeneous wave equation given by

$$\boldsymbol{\nabla}^2\tilde{p} + G(\boldsymbol{x}, t) = \frac{1}{a_0^2}\frac{\partial^2\tilde{p}}{\partial t^2}, \tag{3.35}$$

then we assume the solution form to be

$$\tilde{p}(r,z,t) = \int_0^\infty \int_0^\infty A(\lambda,\gamma,t) J_0(\lambda r) \cos\gamma z \, d\lambda \, d\gamma, \tag{3.36a}$$

$$G(r,z,t) = \int_0^\infty \int_0^\infty R(\lambda,\gamma,t) J_0(\lambda r) \cos\gamma z \, d\lambda \, d\gamma. \tag{3.36b}$$

Note that the assumed form for $\tilde{p}$ automatically satisfies the far-field conditions. Substituting Eqns. (3.36) into Eqn. (3.35), and with a prime denoting a derivative with respect to t, we get

$$A''(\lambda,\gamma,t) + (\lambda^2+\gamma^2)a_0^2 A(\lambda,\gamma,t) = a_0^2 R(\lambda,\gamma,t),$$

whose solution obtained using the Laplace transform method and the initial conditions is

$$A(\lambda,\gamma,t) = \frac{a_0}{\sqrt{\lambda^2+\gamma^2}} \int_0^t \sin\left[a_0\sqrt{\lambda^2+\gamma^2}(t-\tau)\right] R(\lambda,\gamma,\tau)\, d\tau. \tag{3.37}$$

From Eqns. (3.36b), we get

$$R(\lambda,\gamma,t) = \frac{2\lambda}{\pi} \int_0^\infty \int_0^\infty \hat{r} G(\hat{r},\hat{z},t) J_0(\lambda \hat{r}) \cos(\gamma \hat{z})\, d\hat{r}\, d\hat{z}, \tag{3.38}$$

which when substituted into Eqn. (3.37) yields $A(\lambda,\gamma,t)$, and hence $\tilde{p}(r,z,t)$.

If instead of the $z=0$ surface being rigid, if we have $\tilde{p}|_{z=0} = 0$, then $\cos\gamma z$ and $\cos\gamma\hat{z}$ are, respectively, replaced by $\sin\gamma z$ and $\sin\gamma\hat{z}$ in Eqns. (3.36)–(3.38).

3.3 Spherical domain

Consider the axisymmetric case where the acoustic fluid is inside a sphere of radius r_1. If $A(t) = Vi\omega e^{i\omega t}$ and the normal acceleration at $r = r_1$ is $A(t)g(\xi)$, then the periodic steady-state solution obtained using Eqn. (1.95) is

$$\frac{\tilde{p}}{\rho V \omega r_1} = -\frac{i}{r_1} \sum_{n=0}^{\infty} \frac{c_n j_n(kr) P_n(\xi)}{j_n'(kr_1)}, \tag{3.39}$$

where

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} g(\hat{\xi}) P_n(\hat{\xi})\, d\hat{\xi}.$$

As an example, if $g(\xi) = 1$ for $\xi \in [\xi_0, 1]$ (the radially vibrating polar-cap problem), then we get

$$c_n = \frac{P_{n-1}(\xi_0) - P_{n+1}(\xi_0)}{2},$$

which when substituted into Eqns. (3.39) yields the complete solution.

Similarly, for the case where the pressure is prescribed as $G(t)g(\xi)$ where $G(t) = G_0 e^{i\omega t}$, the periodic steady-state solution obtained using Eqn. (1.95) is

$$\frac{\tilde{p}}{G_0} = \sum_{n=0}^{\infty} \frac{c_n j_n(kr) P_n(\xi)}{j_n(kr_1)}, \tag{3.40}$$

where

$$c_n = \frac{2n+1}{2} \int_{-1}^{1} g(\hat{\xi}) P_n(\hat{\xi})\, d\hat{\xi}.$$

Now we turn to the solution of transient problems. The wave equation in spherical coordinates in the presence of spherical symmetry is

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial p_\Delta}{\partial r}\right) = \frac{1}{a_0^2}\frac{\partial^2 p_\Delta}{\partial t^2}.$$

First consider the domain to be the unbounded region outside the sphere of radius r_1 with the normal acceleration $A(t)$ prescribed at the surface. Taking into account that p_Δ is zero as $r \to \infty$, the solution of the Laplace-transformed equation is $\bar{p}_\Delta = c_2 e^{-sr/a_0}/r$. Using the boundary condition at $r = r_1$, we get

$$\begin{aligned} \bar{p}_\Delta &= \frac{\rho a_0 r_1 \bar{A}}{r} \frac{e^{-\frac{s(r-r_1)}{a_0}}}{s + \frac{a_0}{r_1}}, \\ \bar{u}_r &= a_0 r_1 \bar{A} e^{-\frac{s(r-r_1)}{a_0}} \frac{\frac{s}{a_0 r} + \frac{1}{r^2}}{s\left(s + \frac{a_0}{r_1}\right)}. \end{aligned} \tag{3.41}$$

The expression for $\bar{u}_r$ can be written as

$$\bar{u}_r = a_0 r_1 \bar{A} e^{-\frac{s(r-r_1)}{a_0}} \left(\frac{d_1}{s} + \frac{d_2}{s + \frac{a_0}{r_1}}\right),$$

where d_1 and d_2 are found using Eqn. (1.99) as

$$d_1 = \frac{r_1}{a_0 r^2}, \qquad d_2 = \frac{r_1}{a_0 r}\left(\frac{1}{r_1} - \frac{1}{r}\right).$$

Thus, using the convolution theorem, it follows that

$$\frac{p_\Delta}{\rho a_0} = H(t_s)\frac{r_1}{r} \int_0^{t_s} e^{-\frac{a_0}{r_1}(t_s - \tau)} A(\tau)\, d\tau$$

$$
\begin{aligned}
&= H(t_s)\frac{r_1}{r}\int_0^{t_s} e^{-\frac{a_0\tau}{r_1}} A\left(t_s-\tau\right)\,d\tau,\\
u_r &= H(t_s)\int_0^{t_s}\left[\frac{r_1^2}{r^2}+\left(\frac{r_1}{r}-\frac{r_1^2}{r^2}\right)e^{-\frac{a_0}{r_1}(t_s-\tau)}\right]A(\tau)\,d\tau,\\
&= H(t_s)\int_0^{t_s}\left[\frac{r_1^2}{r^2}+\left(\frac{r_1}{r}-\frac{r_1^2}{r^2}\right)e^{-\frac{a_0\tau}{r_1}}\right]A\left(t_s-\tau\right)\,d\tau,
\end{aligned}
$$

where the 'shifted time' t_s is given by

$$
t_s := t - \frac{r-r_1}{a_0}. \tag{3.42}
$$

For the case of an applied step acceleration, i.e., when $A(t) = A_0$, we get $u_r = p_\Delta = 0$ for $t_s < 0$, while for $t_s \geq 0$, we get

$$
\begin{aligned}
\frac{p_\Delta}{\rho A_0 r_1} &= \frac{r_1}{r}\left(1-e^{-\frac{a_0 t_s}{r_1}}\right),\\
\frac{u_r}{A_0} &= \left(\frac{r_1}{r}\right)^2\left(t-\frac{r-r_1}{a_0}e^{-\frac{a_0 t_s}{r_1}}\right).
\end{aligned} \tag{3.43}
$$

Note that at the surface of the sphere, the velocity is $A_0 t$, which is consistent with the acceleration and the zero initial velocity condition.

When $A(t) = V\alpha e^{-\alpha t}$, where α is a positive constant, we get $u_r = p_\Delta = 0$ for $t_s < 0$, while for $t_s \geq 0$, we get (with $k \equiv \alpha/a_0$)

$$
\begin{aligned}
\frac{p_\Delta}{\rho\alpha r_1 V} &= \frac{r_1}{r(1-kr_1)}\left(e^{-\alpha t_s}-e^{-\frac{a_0 t_s}{r_1}}\right),\\
\frac{u_r}{V} &= \frac{r_1^2}{r^2}\left[1-\frac{(1-kr)e^{-\alpha t_s}+k(r-r_1)e^{-\frac{a_0 t_s}{r_1}}}{1-kr_1}\right].
\end{aligned} \tag{3.44}
$$

Consistent with the acceleration and the zero initial velocity condition, the velocity at the surface of the sphere is $V(1-e^{-\alpha t})$. Note that, at a given r, the pressure field first increases sharply after the wave reaches r, attains a peak, and then decays to zero with increasing time.

For the case when $A(t) = V\omega\cos\omega t$, we get $u_r = p_\Delta = 0$ for $t_s < 0$, while for $t_s \geq 0$, we get (with $k \equiv \omega/a_0$)

$$
\begin{aligned}
\frac{p_\Delta}{\rho\omega r_1 V} &= \frac{r_1}{r[1+(kr_1)^2]}\left[\cos\omega t_s + kr_1\sin\omega t_s - e^{-\frac{a_0 t_s}{r_1}}\right],\\
\frac{u_r}{V} &= \frac{r_1^2}{r^2[1+(kr_1)^2]}\left[k(r-r_1)\cos\omega t_s+(1+k^2 r_1 r)\sin\omega t_s - k(r-r_1)e^{-\frac{a_0 t_s}{r_1}}\right].
\end{aligned} \tag{3.45}
$$

Consistent with the acceleration and the zero initial velocity condition, the velocity at the surface of the sphere is $V \sin \omega t$.

Similarly, for the case when $A(t) = V\omega \sin \omega t$, we get $u_r = p = 0$ for $t_s < 0$, while for $t_s \geq 0$, we get

$$\begin{aligned} \frac{p_\Delta}{\rho \omega r_1 V} &= \frac{r_1}{r[1+(kr_1)^2]} \left[\sin \omega t_s - kr_1 \cos \omega t_s + kr_1 e^{-\frac{a_0 t_s}{r_1}} \right], \\ \frac{u_r}{V} &= \frac{r_1^2}{r^2[1+(kr_1)^2]} \left[k(r-r_1) \sin \omega t_s - (1+k^2 r_1 r) \cos \omega t_s + 1 + k^2 r_1^2 + k^2 r_1 (r-r_1) e^{-\frac{a_0 t_s}{r_1}} \right]. \end{aligned} \tag{3.46}$$

Note that the exponential part of the above solution attenuates (with increasing t at a given r) although there is no damping in the model! Note that this happens only in exterior domain problems, as a result of which the steady-state (which includes even periodic steady-state solutions) and transient solutions agree after a long enough time; in interior domain problems, however, the transient part does not get damped out, and the transient and periodic steady-state solutions can be significantly different even after long times. Thus, in exterior domain problems, the boundary at 'infinity' can be considered to be a damper. Consistent with the acceleration and the initial conditions, the velocity at the surface of the sphere is $V(1 - \cos \omega t)$.

If instead of the acceleration, the pressure at $r = r_1$ is specified as $G(t)$, then in place of Eqns. (3.41), we get

$$\begin{aligned} \bar{p}_\Delta &= \frac{\bar{G} r_1}{r} e^{-\frac{s(r-r_1)}{a_0}}, \\ \bar{u}_r &= \frac{\bar{G} r_1}{\rho a_0 r} \left(1 + \frac{a_0}{rs} \right) e^{-\frac{s(r-r_1)}{a_0}}, \end{aligned} \tag{3.47}$$

so that

$$\begin{aligned} p_\Delta &= \frac{r_1}{r} H(t_s) G(t_s), \\ u_r &= \frac{H(t_s)}{\rho} \left[\frac{r_1 G(t_s)}{a_0 r} + \frac{r_1}{r^2} \int_0^{t_s} G(\tau)\, d\tau \right]. \end{aligned} \tag{3.48}$$

As an example, if $G = G_0$, where G_0 is a constant, then $u_r = p_\Delta = 0$ for $t_s < 0$, while for $t_s \geq 0$, we have

$$\begin{aligned} \frac{p_\Delta}{G_0} &= \frac{r_1}{r}, \\ \frac{\rho u_r}{G_0} &= \frac{r_1 t}{r^2} + \frac{r_1^2}{a_0 r^2}. \end{aligned} \tag{3.49}$$

As discussed after Eqn. (3.14), only those $G(t)$ with $G(0) = 0$ are permissible, and so the above solution should be used in superposition with another solution so that this condition is satisfied, e.g., $G(t) = G_0\,(1 - \cos\omega t)$, $G(t) = G_0\,(1 - e^{-\omega t})$ etc.

Now consider the case where the acoustic fluid is inside the spherical domain of radius r_1. If the imposed acceleration on the surface $r = r_1$ is $A(t)$, then the Laplace-transformed solution is

$$\bar{p}_\Delta = \frac{\rho \bar{A} r_1^2 \sin qr}{r\,(\sin qr_1 - qr_1 \cos qr_1)}, \tag{3.50}$$

where $q^2 = -s^2/a_0^2$. Let λ_n, $n = 1, 2, \ldots, \infty$, be the positive roots of

$$\tan x = x.$$

We can write

$$\frac{\bar{p}_\Delta}{\bar{A}} = \frac{d_0}{s^2} + \frac{d_1}{s} + \sum_{n=1}^{\infty} \frac{d_n^{(1)}}{s + \frac{i\lambda_n a_0}{r_1}} + \frac{d_n^{(2)}}{s - \frac{i\lambda_n a_0}{r_1}}, \tag{3.51}$$

where

$$d_0 = \lim_{s\to 0} \frac{s^2 \bar{p}_\Delta}{\bar{A}} = -\frac{3\rho a_0^2}{r_1},$$
$$d_1 = \lim_{s\to 0} \frac{d}{ds}\left(\frac{s^2 \bar{p}_\Delta}{\bar{A}}\right) = 0,$$

and the constants $d_n^{(1)}$ and $d_n^{(2)}$ are found using the Heaviside formula. Using the convolution theorem, we get

$$p_\Delta = \rho a_0 \left[-\frac{3a_0}{r_1}\int_0^t \tau A(t-\tau)\,d\tau - \sum_{n=1}^{\infty} 2\lambda_n d_n \int_0^t \sin\frac{\lambda_n a_0 \tau}{r_1} A(t-\tau)\,d\tau\right],$$

where with $\zeta := r/r_1$,

$$d_n = \frac{\sin(\lambda_n \zeta)}{\lambda_n^2 \zeta \sin\lambda_n}.$$

For the case when $A(t) = A_0$, we get

$$\frac{p_\Delta}{\rho A_0 r_1} = \frac{3}{10} - \frac{\zeta^2}{2} - \frac{3a_0^2 t^2}{2r_1^2} + \sum_{n=1}^{\infty} 2d_n \cos\frac{\lambda_n a_0 t}{r_1}. \tag{3.52}$$

For the case when $A(t) = V\alpha e^{-\alpha t}$, where α is a positive constant, we get (with $k \equiv \alpha/a_0$)

$$\frac{p_\Delta}{\rho a_0 V} = \frac{3(1-\alpha t)}{kr_1} - \frac{kr_1 \sinh(kr_1\zeta)e^{-\alpha t}}{\zeta(kr_1 \cosh kr_1 - \sinh kr_1)}$$

$$+\sum_{n=1}^{\infty}\frac{2\lambda_n d_n\left[\left(\frac{\lambda_n}{kr_1}\right)\cos\frac{\lambda_n a_0 t}{r_1}-\sin\frac{\lambda_n a_0 t}{r_1}\right]}{1+\left(\frac{\lambda_n}{kr_1}\right)^2}.$$

When $A(t)=V\omega\cos\omega t$, we get (with $k\equiv\omega/a_0$)

$$\frac{p_\Delta}{\rho a_0 V}=-\frac{3}{kr_1}-\frac{kr_1\sin(kr_1\zeta)\cos\omega t}{\zeta(kr_1\cos kr_1-\sin kr_1)}+\sum_{n=1}^{\infty}\frac{2\left(\frac{\lambda_n}{kr_1}\right)\lambda_n d_n\cos\frac{\lambda_n a_0 t}{r_1}}{\left(\frac{\lambda_n}{kr_1}\right)^2-1},$$

while for $A(t)=V\omega\sin\omega t$, we get

$$\frac{p_\Delta}{\rho a_0 V}=-\frac{3\omega t}{kr_1}-\frac{kr_1\sin(kr_1\zeta)\sin\omega t}{\zeta(kr_1\cos kr_1-\sin kr_1)}+\sum_{n=1}^{\infty}\frac{2\lambda_n d_n\sin\frac{\lambda_n a_0 t}{r_1}}{\left(\frac{\lambda_n}{kr_1}\right)^2-1}.$$

If the surface $r=r_1$ is rigid, and the initial conditions are $p_\Delta(r,0)=p_0(r)$ and $\dot{p}_\Delta(r,0)=v_0(r)$, then after accounting for the zero root $\lambda_0:=0$, the form of the solution is

$$p_\Delta=a_1+\frac{1}{\zeta}\sum_{n=1}^{\infty}\sin\lambda_n\zeta\left[A_n\cos\left(\frac{\lambda_n a_0 t}{r_1}\right)+B_n\sin\left(\frac{\lambda_n a_0 t}{r_1}\right)\right], \tag{3.53}$$

Imposing the initial conditions, we get

$$a_1+\frac{1}{\zeta}\sum_{n=1}^{\infty}A_n\sin\lambda_n\zeta=p_0(\zeta),$$
$$\frac{1}{\zeta}\sum_{n=1}^{\infty}\frac{\lambda_n B_n\sin\lambda_n\zeta}{\lambda_n}=\frac{r_1 v_0(\zeta)}{a_0},$$

which on using the orthogonality properties given by Eqns. (1.18) yields

$$a_1=3\int_0^1\hat{\zeta}^2 p_0(\hat{\zeta})\,d\hat{\zeta},$$
$$A_n=\frac{2}{\sin^2\lambda_n}\int_0^1\hat{\zeta}p_0(\hat{\zeta})\sin(\lambda_n\hat{\zeta})\,d\hat{\zeta},$$
$$B_n=\frac{2r_1}{a_0\lambda_n\sin^2\lambda_n}\int_0^1\hat{\zeta}v_0(\hat{\zeta})\sin(\lambda_n\hat{\zeta})\,d\hat{\zeta}.$$

As an example, if $v_0(r)=0$, and $p_0(r)=p_0$, then the only nonzero constant is $a_1=p_0$, so that the solution is $p_\Delta=p_0$. The critical role played by the non-series constant a_1 is evident from this example.

If $\tilde{p}|_{t=0} = \dot{\tilde{p}}\big|_{t=0} = 0$, if the surface $r = r_1$ is rigid, and if we consider the non-homogeneous wave equation given by Eqn. (3.35), then we assume the solution form to be

$$\tilde{p}(r,t) = A_0(t) + \frac{1}{\zeta}\sum_{n=1}^{\infty} A_n(t)\sin\lambda_n\zeta, \tag{3.54a}$$

$$G(r,t) = R_0(t) + \frac{1}{\zeta}\sum_{n=1}^{\infty} R_n(t)\sin\lambda_n\zeta. \tag{3.54b}$$

Note that the assumed form for $\tilde{p}$ automatically satisfies the boundary conditions. Substituting Eqns. (3.54) into Eqn. (3.35), and with a prime denoting a derivative with respect to t, we get

$$A_0''(t) = a_0^2 R_0(t),$$
$$A_n''(t) + \left(\frac{\lambda_n a_0}{r_1}\right)^2 A_n(t) = a_0^2 R_n(t),$$

whose solution obtained using the Laplace transform method and the initial conditions is

$$\begin{aligned} A_0(t) &= a_0^2 \int_0^t \tau R_0(t-\tau)\,d\tau, \\ A_n(t) &= \frac{a_0 r_1}{\lambda_n}\int_0^t \sin\left[\frac{a_0\lambda_n(t-\tau)}{r_1}\right] R_n(\tau)\,d\tau. \end{aligned} \tag{3.55}$$

From Eqn. (1.18) with $k = 0$, and Eqn. (3.54b), we get

$$R_0(t) = 3\int_0^1 \hat{\zeta}^2 G(\hat{\zeta},t)\,d\hat{\zeta},$$
$$R_n(t) = \frac{2}{\sin^2\lambda_n}\int_0^1 \hat{\zeta} G(\hat{\zeta},t)\sin(\lambda_n\hat{\zeta})\,d\hat{\zeta},$$

which when substituted into Eqn. (3.55) yields $A_0(t)$ and $A_n(t)$, and hence $\tilde{p}(r,t)$.

If in place of the normal acceleration, the pressure is specified at the surface of the sphere r_1 as $P(t)$, then in place of Eqn. (3.50), we get

$$\bar{p}_\Delta = \frac{(r_1 \sin qr)\bar{P}}{r\sin qr_1}, \tag{3.56}$$

The inverse transform is

$$p_\Delta(r,t) = -\frac{2a_0}{r}\sum_{n=1}^{\infty}(-1)^n \sin\frac{n\pi r}{r_1}\int_0^t \sin(\lambda_n a_0\tau)P(t-\tau)\,d\tau, \tag{3.57}$$

where $\lambda_n = n\pi/r_1$.

For $P(t) = P_0$, we get

$$\frac{p_\Delta(r,t)}{P_0} = 1 + \frac{2}{\zeta}\sum_{n=1}^{\infty}\frac{(-1)^n \sin n\pi\zeta \cos(\lambda_n a_0 t)}{n\pi}.$$

For $P(t) = P_0 \cos\omega t$, we get with $k := \omega/a_0$,

$$\frac{p_\Delta(r,t)}{P_0} = \frac{\sin(kr_1\zeta)\cos\omega t}{\zeta \sin kr_1} + \frac{2}{kr_1\zeta}\sum_{n=1}^{\infty}\frac{(-1)^n\left(\frac{\lambda_n}{k}\right)\sin n\pi\zeta\cos(\lambda_n a_0 t)}{\left(\frac{\lambda_n}{k}\right)^2 - 1},$$

while for $P(t) = P_0 \sin\omega t$, we get

$$\frac{p_\Delta(r,t)}{P_0} = \frac{\sin(kr_1\zeta)\sin\omega t}{\zeta \sin kr_1} + \frac{2}{kr_1\zeta}\sum_{n=1}^{\infty}\frac{(-1)^n \sin n\pi\zeta\sin(\lambda_n a_0 t)}{\left(\frac{\lambda_n}{k}\right)^2 - 1}.$$

Finally, for $P(t) = P_0 e^{-\alpha t}$, we get with $k := \alpha/a_0$,

$$\frac{p_\Delta(r,t)}{P_0} = \frac{\sinh(kr_1\zeta)e^{-\alpha t}}{\zeta \sinh kr_1} + \frac{2}{kr_1\zeta}\sum_{n=1}^{\infty}\frac{(-1)^n \sin n\pi\zeta}{\left(\frac{\lambda_n}{k}\right)^2 + 1}\left[\frac{\lambda_n}{k}\cos(\lambda_n a_0 t) - \sin(\lambda_n a_0 t)\right].$$

Note that $p_\Delta(r,t) = 0$ for $t \le (r_1 - r)/a_0$ in all the above solutions. As noted in the discussion following Eqn. (3.14), only those $P(t)$ with $P(0) = 0$ are permissible, and so the above solutions should be used in superposition with each other so that this condition is satisfied, e.g., $P(t) = P_0\,(1 - \cos\omega t)$, $P(t) = P_0\,(1 - e^{-\alpha t})$ etc.

If $P(t) = 0$ and the initial conditions are $p_\Delta(r,0) = p_0(r)$ and $\dot{p}_\Delta(r,0) = v_0(r)$, then the form of the solution is given by

$$p_\Delta = \frac{1}{\zeta}\sum_{n=1}^{\infty}\sin n\pi\zeta\left[A_n\cos(\lambda_n a_0 t) + B_n \sin(\lambda_n a_0 t)\right]. \tag{3.58}$$

Imposing the initial conditions, we get

$$\frac{1}{\zeta}\sum_{n=1}^{\infty}A_n\sin(n\pi\zeta) = p_0(\zeta),$$
$$\frac{1}{\zeta}\sum_{n=1}^{\infty}\lambda_n B_n\sin(n\pi\zeta) = \frac{r_1 v_0(\zeta)}{a_0},$$

which on using the orthogonality properties of the sine functions leads to

$$A_n = 2\int_0^1 \hat{\zeta} p_0(\hat{\zeta})\sin(n\pi\hat{\zeta})\,d\hat{\zeta},$$

$$B_n = \frac{2r_1}{a_0\lambda_n}\int_0^1 \hat{\zeta} v_0(\hat{\zeta})\sin(n\pi\hat{\zeta})\,d\hat{\zeta}.$$

As an example, if $v_0(r) = 0$ and $p_0(r) = p_0$, then we have $B_n = 0$ and $A_n = 2(-1)^{n+1}p_0/(n\pi)$, so that

$$\frac{p_\Delta}{p_0} = \frac{2}{\zeta}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}\sin n\pi\zeta\cos(\lambda_n a_0 t)}{n\pi}. \tag{3.59}$$

If $\tilde{p}\big|_{t=0} = \dot{\tilde{p}}\big|_{t=0} = P(t) = 0$, and if we consider the non-homogeneous wave equation given by Eqn. (3.35), then we assume the solution form to be

$$\tilde{p}(r,t) = \frac{1}{\zeta}\sum_{n=1}^{\infty} A_n(t)\sin n\pi\zeta, \tag{3.60a}$$

$$G(r,t) = \frac{1}{\zeta}\sum_{n=1}^{\infty} R_n(t)\sin n\pi\zeta. \tag{3.60b}$$

Note that the assumed form for $\tilde{p}$ automatically satisfies the boundary conditions. Substituting Eqns. (3.60) into Eqn. (3.35), and with a prime denoting a derivative with respect to t, we get

$$A_n''(t) + \left(\frac{n\pi a_0}{r_1}\right)^2 A_n(t) = a_0^2 R_n(t),$$

whose solution obtained using the Laplace transform method and the initial conditions is

$$A_n(t) = \frac{a_0 r_1}{n\pi}\int_0^t \sin\left[\frac{a_0 n\pi(t-\tau)}{r_1}\right] R_n(\tau)\,d\tau. \tag{3.61}$$

From Eqns. (3.60b), and the orthogonality of the sine functions, we get

$$R_n(t) = 2\int_0^1 \hat{\zeta} G(\hat{\zeta},t)\sin(n\pi\hat{\zeta})\,d\hat{\zeta},$$

which when substituted into Eqn. (3.61) yields $A_n(t)$, and hence $\tilde{p}(r,t)$.

Appendix A

Error and complementary error functions

Let $x \in \Re$, $a \geq 0$, $\alpha \geq 0$, and $\gamma > 0$. Further, let erf(.) and $\mathrm{erfc}(x) = 1 - \mathrm{erf}(x)$ denote the error and complementary error functions defined by

$$\mathrm{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-\lambda^2}\, d\lambda, \tag{A.1a}$$

$$\mathrm{erfc}(x) := \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-\lambda^2}\, d\lambda. \tag{A.1b}$$

From the definition, it follows that $\mathrm{erf}(0) = 0$ and $\mathrm{erf}(\infty) = 1$. Then

$$\int_x^\infty \frac{e^{-(\gamma\lambda)^2}\, d\lambda}{\lambda^2} = \frac{e^{-(\gamma x)^2}}{x} - \gamma\sqrt{\pi}\,\mathrm{erfc}(\gamma x), \tag{A.2a}$$

$$\int_0^\infty \cos(\lambda x) e^{-\gamma\lambda}\, d\lambda = \frac{\gamma}{x^2 + \gamma^2}, \tag{A.2b}$$

$$\int_0^\infty \cos(\lambda x) e^{-(\gamma\lambda)^2}\, d\lambda = \frac{\sqrt{\pi}}{2\gamma} e^{-\frac{x^2}{(2\gamma)^2}}, \tag{A.2c}$$

$$\int_0^\infty e^{-(\gamma\lambda)^2} \cos\left(a - \frac{\alpha^2}{2\lambda^2}\right) d\lambda = \int_0^\infty \frac{e^{-\gamma^2/\tau}}{2\tau^{3/2}} \cos\left(a - \frac{\alpha^2\tau}{2}\right) d\tau = \frac{\sqrt{\pi}}{2\gamma} e^{-\gamma\alpha} \cos\left(a - \gamma\alpha\right), \tag{A.2d}$$

$$\int_0^\infty e^{-(\gamma\lambda)^2} \sin\left(a - \frac{\alpha^2}{2\lambda^2}\right) d\lambda = \int_0^\infty \frac{e^{-\gamma^2/\tau}}{2\tau^{3/2}} \sin\left(a - \frac{\alpha^2\tau}{2}\right) d\tau = \frac{\sqrt{\pi}}{2\gamma} e^{-\gamma\alpha} \sin\left(a - \gamma\alpha\right), \tag{A.2e}$$

$$\int_0^\infty \frac{e^{-(\gamma\lambda)^2}}{\lambda^2} \cos\left(a - \frac{\alpha^2}{2\lambda^2}\right) d\lambda = \int_0^\infty \frac{e^{-\gamma^2/\tau}}{2\sqrt{\tau}} \cos\left(a - \frac{\alpha^2\tau}{2}\right) d\tau$$

$$= \frac{\sqrt{\pi}}{2\alpha} e^{-\gamma\alpha} \left[\sin\left(a - \gamma\alpha\right) + \cos\left(a - \gamma\alpha\right)\right], \tag{A.2f}$$

$$\int_0^\infty \frac{e^{-(\gamma\lambda)^2}}{\lambda^2} \sin\left(a - \frac{\alpha^2}{2\lambda^2}\right) d\lambda = \int_0^\infty \frac{e^{-\gamma^2/\tau}}{2\sqrt{\tau}} \sin\left(a - \frac{\alpha^2\tau}{2}\right) d\tau$$

$$= \frac{\sqrt{\pi}}{2\alpha} e^{-\gamma\alpha} \left[\sin\left(a - \gamma\alpha\right) - \cos\left(a - \gamma\alpha\right)\right], \tag{A.2g}$$

$$\int_0^\infty e^{-[\gamma(\lambda+x)]^2} \, d\lambda = \frac{\sqrt{\pi}}{2\gamma} \operatorname{erfc}(\gamma x), \tag{A.2h}$$

$$\int_0^\infty \lambda e^{-[\gamma(\lambda+x)]^2} \, d\lambda = \frac{1}{2\gamma^2} \left[e^{-(\gamma x)^2} - \sqrt{\pi}\gamma x \operatorname{erfc}(\gamma x)\right], \tag{A.2i}$$

$$\int_0^t \frac{e^{-a^2\tau} \, d\tau}{\sqrt{\tau}} = \frac{1}{a}\sqrt{\pi} \operatorname{erf}(a\sqrt{t}), \tag{A.2j}$$

$$\int_0^t \frac{e^{-b^2/\tau} \, d\tau}{\sqrt{\tau}} = 2\sqrt{t} e^{-b^2/t} - 2b\sqrt{\pi} \operatorname{erfc} \frac{b}{\sqrt{t}}, \quad (b \geq 0), \tag{A.2k}$$

(A.2l)

Appendix B

Table of Laplace Transforms

With $a > 0$, a table of Laplace transforms is given in Table B.1.

$\mathcal{L}[f(t)]$	$f(t)$
$\frac{1}{s^n}$, $n = 1, 2, \ldots$	$\frac{t^{n-1}}{(n-1)!}$
$\frac{1}{s+a}$	e^{-at}
$\frac{\omega}{s^2+\omega^2}$	$\sin \omega t$
$\frac{s}{s^2+\omega^2}$	$\cos \omega t$
$\frac{e^{-as}}{s}$	$H(t-a)$
$e^{-a\sqrt{s}}$	$\frac{ae^{-a^2/(4t)}}{2\sqrt{\pi t^3}}$
$\frac{e^{-a\sqrt{s}}}{\sqrt{s}}$	$\frac{e^{-a^2/(4t)}}{\sqrt{\pi t}}$
$\frac{e^{-a\sqrt{s}}}{s}$	$\operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right)$
$\frac{e^{-a\sqrt{s}}}{s^2}$	$\left(t + \frac{a^2}{2}\right) \operatorname{erfc}\left(\frac{a}{2\sqrt{t}}\right) - \frac{a\sqrt{t}e^{-a^2/(4t)}}{\sqrt{\pi}}$
$\frac{1}{s\sqrt{s+a}}$	$\frac{\operatorname{erf}\sqrt{at}}{\sqrt{a}}$
$\frac{1}{\sqrt{s^2+a^2}}$	$J_0(at)$

Table B.1: Table of Laplace transforms.

Bibliography

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