ME242: Assignment 1

Due: 1/9/14

Using indicial or direct notation, prove the following:

1. Show that

$$(\boldsymbol{a} \times \boldsymbol{b}) \cdot (\boldsymbol{c} \times \boldsymbol{d}) = (\boldsymbol{a} \cdot \boldsymbol{c})(\boldsymbol{b} \cdot \boldsymbol{d}) - (\boldsymbol{a} \cdot \boldsymbol{d})(\boldsymbol{b} \cdot \boldsymbol{c}),$$
$$\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{w},$$
$$(\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w} = (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{w})\boldsymbol{u}.$$

2. Show that

$$(\boldsymbol{a} \otimes \boldsymbol{b})^T = \boldsymbol{b} \otimes \boldsymbol{a},$$
$$(\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{c} \otimes \boldsymbol{d}) = (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a} \otimes \boldsymbol{d},$$
$$(\boldsymbol{a} \otimes \boldsymbol{b}) : (\boldsymbol{u} \otimes \boldsymbol{v}) = (\boldsymbol{a} \cdot \boldsymbol{u})(\boldsymbol{b} \cdot \boldsymbol{v}),$$
$$\boldsymbol{T}(\boldsymbol{a} \otimes \boldsymbol{b}) = (\boldsymbol{T}\boldsymbol{a}) \otimes \boldsymbol{b},$$
$$(\boldsymbol{a} \otimes \boldsymbol{b})\boldsymbol{T} = \boldsymbol{a} \otimes (\boldsymbol{T}^T\boldsymbol{b}),$$
$$\boldsymbol{T} : (\boldsymbol{a} \otimes \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{T}\boldsymbol{b}.$$

   Use any of the above equations to show that for a symmetric $\boldsymbol{S}$, the spectral decompositions of $\boldsymbol{S}^2$ and $\boldsymbol{S}^{-1}$ (when it exists) are

$$\boldsymbol{S}^2 = \lambda_1^2 \boldsymbol{e}_1^* \otimes \boldsymbol{e}_1^* + \lambda_2^2 \boldsymbol{e}_2^* \otimes \boldsymbol{e}_2^* + \lambda_3^2 \boldsymbol{e}_3^* \otimes \boldsymbol{e}_3^*,$$
$$\boldsymbol{S}^{-1} = \lambda_1^{-1} \boldsymbol{e}_1^* \otimes \boldsymbol{e}_1^* + \lambda_2^{-1} \boldsymbol{e}_2^* \otimes \boldsymbol{e}_2^* + \lambda_3^{-1} \boldsymbol{e}_3^* \otimes \boldsymbol{e}_3^*.$$

3. Using any of the above results, show that

$$\boldsymbol{Q} = \boldsymbol{e}_i \otimes \bar{\boldsymbol{e}}_i,$$

   is an orthogonal tensor. Show that the components of $\boldsymbol{Q}$ are $Q_{ij} = \bar{\boldsymbol{e}}_i \cdot \boldsymbol{e}_j$.

4. Show that

$$\boldsymbol{R} : (\boldsymbol{S}\boldsymbol{T}) = (\boldsymbol{S}^T\boldsymbol{R}) : \boldsymbol{T} = (\boldsymbol{R}\boldsymbol{T}^T) : \boldsymbol{S} = (\boldsymbol{T}\boldsymbol{R}^T) : \boldsymbol{S}^T,$$

5. The components of a symmetric tensor corresponding to a given basis are given by

$$\boldsymbol{\tau} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & -3\sqrt{3} \\ 0 & -3\sqrt{3} & -5 \end{bmatrix}$$

   Find the principal values and principal directions of this tensor, and find the components of the tensor with respect to the principal basis (*without* using the results of the theorem).

6. If $\boldsymbol{S}$ and $\boldsymbol{W}$ are symmetric and antisymmetric tensors, respectively, and $\boldsymbol{T}$ is an arbitrary second-order tensor prove that

$$\begin{aligned}\boldsymbol{S} : \boldsymbol{T} = \boldsymbol{S} : \boldsymbol{T}^T = \boldsymbol{S} : \left[\frac{1}{2}(\boldsymbol{T}^T + \boldsymbol{T})\right] \\ \boldsymbol{W} : \boldsymbol{T} = -\boldsymbol{W} : \boldsymbol{T}^T = \boldsymbol{W} : \left[\frac{1}{2}(\boldsymbol{T} - \boldsymbol{T}^T)\right] \\ \boldsymbol{S} : \boldsymbol{W} = 0.\end{aligned} \tag{1}$$

7. Find the principal invariants of a skew-symmetric tensor, $\boldsymbol{W}$, and deduce that $\boldsymbol{W}$ has only one real eigenvalue.

8. Show that $\boldsymbol{\nabla} \cdot [(\boldsymbol{\nabla u})\boldsymbol{v}] = (\boldsymbol{\nabla u})^T : \boldsymbol{\nabla v} + \boldsymbol{v} \cdot [\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u})]$.

9. Show that $\boldsymbol{\nabla} \cdot (\boldsymbol{u} \times \boldsymbol{v}) = \boldsymbol{v} \cdot (\boldsymbol{\nabla} \times \boldsymbol{u}) - \boldsymbol{u} \cdot (\boldsymbol{\nabla} \times \boldsymbol{v})$.

10. Show that

$$\begin{aligned}\boldsymbol{\nabla} \cdot (\phi \boldsymbol{v}) = \phi(\boldsymbol{\nabla} \cdot \boldsymbol{v}) + \boldsymbol{v} \cdot (\boldsymbol{\nabla} \phi), \\ \boldsymbol{\nabla} \cdot (\boldsymbol{T}^T \boldsymbol{v}) = \boldsymbol{T} : \boldsymbol{\nabla v} + \boldsymbol{v} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{T}).\end{aligned}$$