ME242: Assignment 2

Due: 12/9/14

Using indicial or direct notation, prove the following:

1. Show that

$$\boldsymbol{\nabla} \cdot (\phi \boldsymbol{T}) = \phi \boldsymbol{\nabla} \cdot \boldsymbol{T} + \boldsymbol{T} \boldsymbol{\nabla} \phi,$$
$$\boldsymbol{\nabla}(\phi \boldsymbol{T} \boldsymbol{v}) = \phi \boldsymbol{\nabla}(\boldsymbol{T}\boldsymbol{v}) + (\boldsymbol{T}\boldsymbol{v}) \otimes \boldsymbol{\nabla} \phi,$$
$$\boldsymbol{\nabla}^2(\boldsymbol{u} \cdot \boldsymbol{v}) = \boldsymbol{u} \cdot \boldsymbol{\nabla}^2 \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{\nabla}^2 \boldsymbol{u} + 2\boldsymbol{\nabla}\boldsymbol{u} : \boldsymbol{\nabla}\boldsymbol{v}.$$

2. If $\boldsymbol{u} = \boldsymbol{x}/\left|\boldsymbol{x}\right|^3$, prove that $\boldsymbol{\nabla} \times \boldsymbol{u} = \boldsymbol{0}$ and $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$.

3. If $\boldsymbol{x}$ is the position vector of a point, and $\boldsymbol{t}$ is the axial vector of $(\boldsymbol{T} - \boldsymbol{T}^T)$, show that

$$\int_V \left[\boldsymbol{x} \times (\boldsymbol{\nabla} \cdot \boldsymbol{T}) + \boldsymbol{t}\right] dV = \int_S \boldsymbol{x} \times (\boldsymbol{T}\boldsymbol{n})\, dS.$$

4. If $\boldsymbol{Q}$ is an orthogonal tensor, show that $\dot{\boldsymbol{Q}}\boldsymbol{Q}^T$ is a skew-symmetric tensor.

5. For the motion described by

$$x_1 = X_1 + \gamma X_2,$$
$$x_2 = X_2,$$
$$x_3 = X_3.$$

evaluate the quantities, $\boldsymbol{F}$, $\boldsymbol{E}$, $\bar{\boldsymbol{E}}$, $\boldsymbol{\epsilon}$, $\tilde{\boldsymbol{v}}$ (Lagrangian velocity description), $\boldsymbol{v}$ (Eulerian velocity description) and $\boldsymbol{\nabla}_{\boldsymbol{x}}\boldsymbol{v}$. Assume γ to be a function of time alone. The above motion is known as pure shear.

6. For the two-dimensional velocity field given by

$$v_x = x,$$
$$v_y = (1 + t^2)y$$

find an expression for the acceleration of a particle whose initial position is $\boldsymbol{X}$ using both the Lagrangian and Eulerian approaches. (Hint: In order to use the Lagrangian approach, you will first have to determine the mapping $\boldsymbol{\chi}$). Verify that both approaches yield the same value of the acceleration at any time, say $t = t_0$.