ME242: Assignment 3

Due: 22/9/14

1. Let $\boldsymbol{L} = \boldsymbol{\nabla}_{\boldsymbol{x}} \boldsymbol{v}$ be the velocity gradient. Show that
$$\frac{D\boldsymbol{F}}{Dt} := \left(\frac{\partial \boldsymbol{F}}{\partial t}\right)_{\boldsymbol{X}} = \boldsymbol{L}\boldsymbol{F},$$
$$\frac{D\boldsymbol{F}^{-1}}{Dt} := \left(\frac{\partial \boldsymbol{F}^{-1}}{\partial t}\right)_{\boldsymbol{X}} = -\boldsymbol{F}^{-1}\boldsymbol{L}.$$
Using the above relations and the relations $DJ/Dt = J(\boldsymbol{\nabla} \cdot \boldsymbol{v})$ and $\boldsymbol{n}\, dS = J\boldsymbol{F}^{-T}\boldsymbol{n}^0\, dS_0$, prove that
$$\frac{d}{dt}\int_{S(t)} \boldsymbol{H}\boldsymbol{n}\, dS = \int_{S(t)} \left[\frac{D\boldsymbol{H}}{Dt} + (\boldsymbol{\nabla} \cdot \boldsymbol{v})\boldsymbol{H} - \boldsymbol{H}(\boldsymbol{\nabla}\boldsymbol{v})^T\right] \boldsymbol{n}\, dS,$$
where $\boldsymbol{H}(\boldsymbol{x}, t)$ is a second-order tensor, and $\boldsymbol{n}$ and $\boldsymbol{n}^0$ are the normals to $S(t)$ and S_0 in the deformed and undeformed configurations, respectively.

2. The stress tensor at the origin is given by
$$\boldsymbol{\tau} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$
Find the normal stress at the origin on the surface $x^2 + y^2 + z^2 + 2x + 4y + 4z = 0$. Find the principal stresses and the equation of the plane through the origin on which the normal stress is a maximum.

3. The traction vectors on three planes at a point are
$$\boldsymbol{t}(\boldsymbol{n}) = \boldsymbol{e}_1 + 2\boldsymbol{e}_2 + 3\boldsymbol{e}_3 \quad \text{for } \boldsymbol{n} = \boldsymbol{e}_1,$$
$$\boldsymbol{t}(\boldsymbol{n}) = 2\sqrt{3}\boldsymbol{e}_1 + 2\sqrt{3}\boldsymbol{e}_2 \quad \text{for } \boldsymbol{n} = \frac{1}{\sqrt{3}}(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3),$$
$$\boldsymbol{t}(\boldsymbol{n}) = 2(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3) \quad \text{for } \boldsymbol{n} = \boldsymbol{e}_2.$$
Find the stress tensor components with respect to the canonical basis $(\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3)$.

4. A body of arbitrary shape is immersed in a fluid which exerts a uniform pressure p on the surface. Neglecting body forces, show that the stress distribution $\tau_{xx} = \tau_{yy} = \tau_{zz} = -p$ satisfies the equations of equilibrium and the traction boundary condition at the surface.

5. A prismatic bar (i.e., a bar of uniform cross-section (not necessarily circular)) of length L hangs from a fixed support. If the origin is located at the centroid of the fixed cross-section with the z-axis pointing downwards, show that the stress distribution $\tau_{zz} = \rho g(L - z)$, other stress components zero, satisfies the equations of equilibrium and the traction boundary conditions at the free surfaces of the bar.