Indian Institute of Science

ME 242: Final Exam

Date: 3/12/02.
Duration: 9.00 a.m.–12.00 p.m.
Maximum Marks: 100

1. By differentiating the strain energy density function for a linear isotropic material given by (25)
$$W = \frac{1}{2}\lambda(\operatorname{tr}\boldsymbol{\epsilon})^2 + \mu\boldsymbol{\epsilon} : \boldsymbol{\epsilon},$$
find an expression for the stress. Substituting this expression for the stress in the linear balance momentum equations given by
$$\rho\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \boldsymbol{\nabla}\cdot\boldsymbol{\tau} + \rho\boldsymbol{b},$$
show that we get the following Navier equations of elasticity:
$$\rho\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \mu\boldsymbol{\nabla}^2\boldsymbol{u} + (\lambda+\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) + \rho\boldsymbol{b}.$$
Show that when $\boldsymbol{b} = \boldsymbol{0}$, the displacements
$$u_i = \frac{\partial\phi}{\partial x_i} + \epsilon_{irs}\frac{\partial A_s}{\partial x_r},$$
satisfy the above equations if
$$\boldsymbol{\nabla}^2\phi = \frac{\rho}{(\lambda+2\mu)}\frac{\partial^2\phi}{\partial t^2}; \quad \boldsymbol{\nabla}^2 A_k = \frac{\rho}{\mu}\frac{\partial^2 A_k}{\partial t^2}.$$

2. Determine the outer radius b of the spinning compound disk to give $\sigma_r = 0$ at the junction $r = 1$. Also, determine the stresses at $r = 0, 1, b$, and the displacement at $r = 1$ (junction). Present your results in a tabular form as shown below. (25)

r (m)	Material	σ_r (MPa)	σ_θ (MPa)	Displacement (mm)
0	A	?	?	0
1	A	0	?	?
1	B	0	?	?
b	B	?	?	0

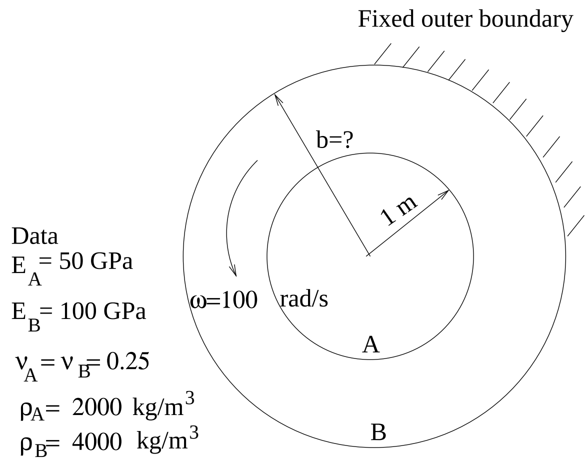

Figure 1: Problem 2

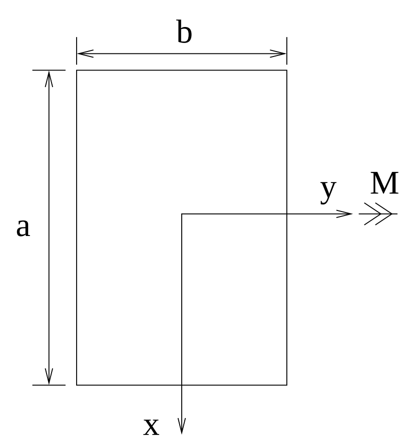

Figure 2: Problem 3

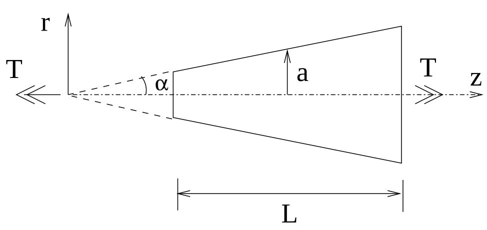

Figure 3: Problem 4

3. A prismatic beam of rectangular cross-section is subjected to a bending moment M_y at its ends (see Fig. 2). It is proposed that the stress distribution instead of being given by $\tau_{zz} = -Ex/R_x$ is given by (20)

$$\tau_{zz} = -\frac{Ex^3}{R_x}, \tag{1}$$

with the other stress components zero.

   (a) Verify that the equlibrium equations (under zero body forces) are satisfied.

   (b) Verify that the lateral surfaces are traction free.

   (c) Verify that the forces on the end surfaces are zero.

   (d) Find an expression for R_x in terms of M_y.

   This expression for R_x when substituted in Eqn. (1) yields a solution (it is a solution since, as shown, it satisfies the governing equations and boundary conditions) that is different from the solution that we studied in class ($\tau_{zz} = -Ex/R_x$). However, we have also proved that the stress solution in the linear elasticity context is unique. How does one resolve this contradiction? Justify mathematically.

4. A conical shaft of length L is subjected to end twisting moments M_t as shown in Fig. 3. The stress distribution which satisfies the equilibrium equations in the bar is given by (30)

$$\begin{aligned}\tau_{r\theta} &= -c\left[\frac{1}{(r^2+z^2)^{3/2}} - \frac{z^2}{(r^2+z^2)^{5/2}}\right],\\ \tau_{\theta z} &= -\frac{crz}{(r^2+z^2)^{5/2}},\end{aligned}$$

with the other stress components zero.

   (a) Find the expression for the unit normal at any point on the lateral surface (i.e., at $r = R$) in terms of $\cos\alpha$ and $\sin\alpha$ (Hint: $n_\theta = 0$), and the expressions for the stresses at the lateral surface in terms of $\cos\alpha$, $\sin\alpha$ and z. In deriving the latter expressions, make use of the fact that (see figure)

$$R = z\tan\alpha.$$

Using the Cauchy principle, verify that the lateral surfaces are traction free.

(b) Find the constant c as a function of the twisting moment M_t and $\cos\alpha$ (Hint: Again use the formulae $R = z\tan\alpha$.)

Some relevant formulae

The equations of compatibility are

$$
\begin{aligned}
\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} &= \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}, \\
\frac{\partial^2 \epsilon_{yy}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial y^2} &= \frac{\partial^2 \gamma_{yz}}{\partial y \partial z}, \\
\frac{\partial^2 \epsilon_{zz}}{\partial x^2} + \frac{\partial^2 \epsilon_{xx}}{\partial z^2} &= \frac{\partial^2 \gamma_{zx}}{\partial z \partial x}, \\
2\frac{\partial^2 \epsilon_{xx}}{\partial y \partial z} &= \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right), \\
2\frac{\partial^2 \epsilon_{yy}}{\partial z \partial x} &= \frac{\partial}{\partial y}\left(-\frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{yx}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x}\right), \\
2\frac{\partial^2 \epsilon_{zz}}{\partial x \partial y} &= \frac{\partial}{\partial z}\left(-\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{zy}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y}\right).
\end{aligned}
$$

Integration formula

$$
\int_0^R \frac{r^3 z\, dr}{(r^2+z^2)^{5/2}} = \frac{1}{3}\left[2 - \frac{2z^3 + 3R^2 z}{(z^2+R^2)^{3/2}}\right].
$$