Indian Institute of Science

ME 242: Final Exam

Date: 6/12/12.
Duration: 9.30 a.m.–12.30 p.m.
Maximum Marks: 100

1. Let $\boldsymbol{g}(\boldsymbol{x}, t)$ and $\boldsymbol{h}(\boldsymbol{x}, t)$ be vector-valued fields related by (15)

$$\boldsymbol{h}(\boldsymbol{x}, t) = e^{k_0|\boldsymbol{x}|-ct}\boldsymbol{g}(\boldsymbol{x}, t),$$

where k_0 and c are constants, and $|\boldsymbol{x}| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}}$. By substituting the above equation into the governing equation

$$\boldsymbol{\nabla} \times \boldsymbol{h} + \frac{1}{c}\frac{\partial \boldsymbol{h}}{\partial t} = \boldsymbol{0},$$

find the governing equation for $\boldsymbol{g}(\boldsymbol{x}, t)$ in *tensorial form.* (Hint: This governing equation should not involve $e^{k_0|\boldsymbol{x}|-ct}$).

2. A large plate of isotropic elastic material contains a circular hole of radius a, and is in a state of pure shear s (i.e., $\tau_{xy} = s$) at large distances from the hole, as shown in Fig. 1. Assuming the stresses to be given by (20)

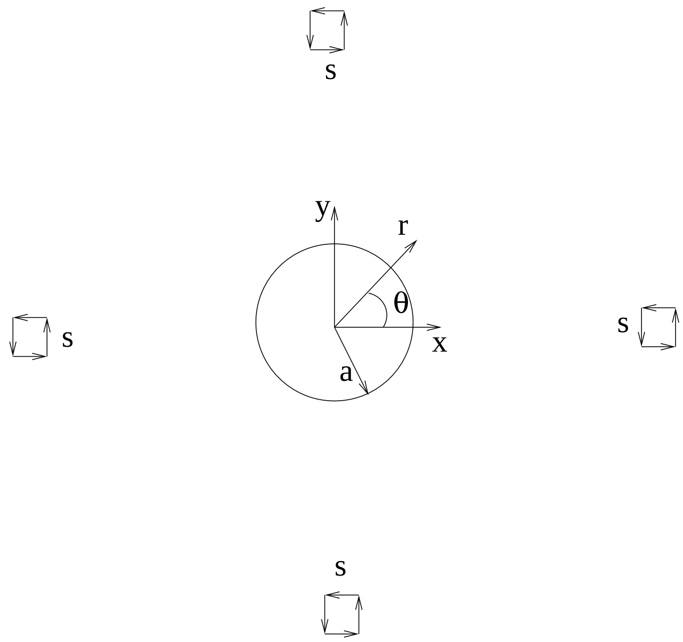

Figure 1: Plate with a circular hole with a state of pure shear at infinity

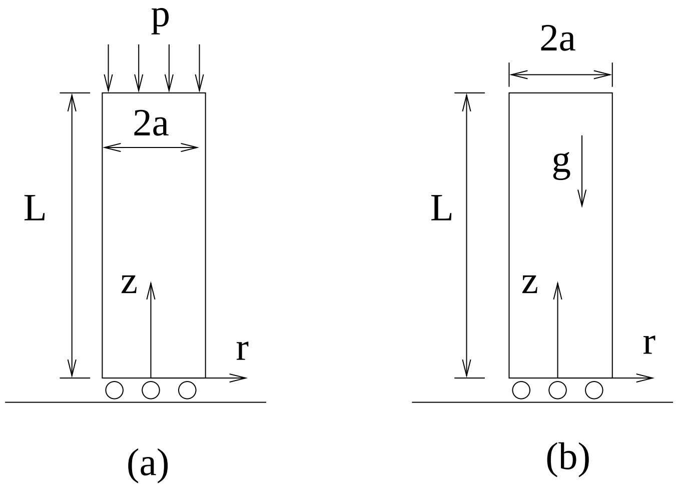

Figure 2: A circular cylinder standing on a frictionless surface: (a) Loaded by uniform traction p alone (b) Loaded by gravity alone.

$$\begin{aligned}
\tau_{rr} &= \frac{A}{r^2} - 2\left(\frac{3C}{r^4} + \frac{2B}{r^2} + D\right)\sin 2\theta, \\
\tau_{\theta\theta} &= -\frac{A}{r^2} + 2\left(D + \frac{3C}{r^4}\right)\sin 2\theta, \\
\tau_{r\theta} &= 2\left(\frac{3C}{r^4} + \frac{B}{r^2} - D\right)\cos 2\theta,
\end{aligned}$$

and assuming the edge of the hole to be traction free, find A, B, C and D.

3. Assume that the constitutive relation is given by $\tau_{ij} = \mathsf{C}_{ijkl}\epsilon_{kl}$, where $\mathbf{C}$ is symmetric, i.e., $\mathsf{C}_{ijkl} = \mathsf{C}_{klij}$. Let $(\boldsymbol{u}^{(1)}, \boldsymbol{\epsilon}^{(1)}, \boldsymbol{\tau}^{(1)}, \boldsymbol{t}^{(1)})$ and $(\boldsymbol{u}^{(2)}, \boldsymbol{\epsilon}^{(2)}, \boldsymbol{\tau}^{(2)}, \boldsymbol{t}^{(2)})$, where $\boldsymbol{\epsilon}^{(1)} \equiv \boldsymbol{\epsilon}(\boldsymbol{u}^{(1)})$ and $\boldsymbol{\epsilon}^{(2)} \equiv \boldsymbol{\epsilon}(\boldsymbol{u}^{(2)})$, denote the static displacement, strain, stress and traction fields (which satisfy all the equations of equilibrium and boundary conditions) under two different loading conditions on the *same domain* V with surface S. Let the body forces in these two loading conditions be denoted by $\boldsymbol{b}^{(1)}$ and $\boldsymbol{b}^{(2)}$, respectively. (35)

   (a) Find a relation between $\int_S \boldsymbol{t}^{(1)} \cdot \boldsymbol{u}^{(2)}\, dS + \int_V \rho \boldsymbol{b}^{(1)} \cdot \boldsymbol{u}^{(2)}\, dV$ and $\int_V \boldsymbol{\tau}^{(1)} : \boldsymbol{\epsilon}^{(2)}\, dV$.

   (b) Interchange the indices (1) and (2), and write the corresponding relation between $\int_S \boldsymbol{t}^{(2)} \cdot \boldsymbol{u}^{(1)}\, dS + \int_V \rho \boldsymbol{b}^{(2)} \cdot \boldsymbol{u}^{(1)}\, dV$ and $\int_V \boldsymbol{\tau}^{(2)} : \boldsymbol{\epsilon}^{(1)}\, dV$.

   (c) Using the symmetry of $\mathbf{C}$, find a relation between $\int_V \boldsymbol{\tau}^{(1)} : \boldsymbol{\epsilon}^{(2)}\, dV$, and $\int_V \boldsymbol{\tau}^{(2)} : \boldsymbol{\epsilon}^{(1)}\, dV$, and hence between $\int_S \boldsymbol{t}^{(1)} \cdot \boldsymbol{u}^{(2)}\, dS + \int_V \rho \boldsymbol{b}^{(1)} \cdot \boldsymbol{u}^{(2)}\, dV$ and $\int_S \boldsymbol{t}^{(2)} \cdot \boldsymbol{u}^{(1)}\, dS + \int_V \rho \boldsymbol{b}^{(2)} \cdot \boldsymbol{u}^{(1)}\, dV$.

   (d) Consider a circular cylinder of radius a and length L standing on a frictionless surface. In one case, it is loaded by a uniform traction p on its top surface (see Fig. 2a). In the other case, it is loaded by gravity loading alone (see Fig. 2b). By assuming the displacement field to be

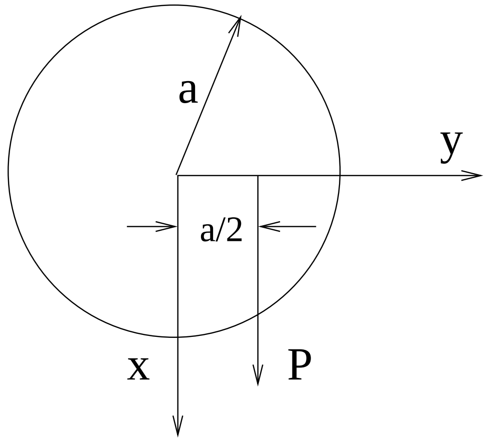

Figure 3: Beam of circular cross section loaded by a terminal load.

of the form $u_r = c_1 r$ and $u_z = c_2 z$ for the loading shown in Fig. 2a, find the displacements and stresses. Using this result and the result in part (c) above, find $\int_A u_z|_{z=L}$ for the gravity loaded cylinder, where A denotes the top circular surface of the cylinder (Hint: *Do not* try to find the exact solution for the loading in Fig. 2b).

4. A beam of circular cross section is loaded by a distribution of tractions on its end face that results in a transverse load P at a distance of $a/2$ from the center as shown in Fig. 3. Find the angle of twist α, and the stresses $\tau_{xz}|_{x=y=0}$ and $\tau_{yz}|_{x=y=0}$. *Do not* use strength of material formulae to obtain your answers. Derive all the results based on a theory of elasticity approach. You may directly use the following formulae, use the polar coordinate system for carrying out integrations, and guess the value of I_{xy}. (30)

$$\kappa_x = \frac{I_{xx}W_x + I_{xy}W_y}{E(I_{xx}I_{yy} - I_{xy}^2)}, \quad \kappa_y = \frac{I_{xy}W_x + I_{yy}W_y}{E(I_{xx}I_{yy} - I_{xy}^2)}; \quad I_{xx} = I_{yy} = \frac{\pi a^4}{4}.$$

$$\tau_{xz} = \frac{\partial \phi}{\partial y} + f(y) - \frac{1}{2}E\kappa_x x^2,$$

$$\tau_{yz} = -\frac{\partial \phi}{\partial x} - g(x) - \frac{1}{2}E\kappa_y y^2,$$

$$\left[\frac{1}{2}E\kappa_x x^2 - f(y)\right]\frac{dy}{ds} = 0, \quad \left[\frac{1}{2}E\kappa_y y^2 + g(x)\right]\frac{dx}{ds} = 0,$$

$$\boldsymbol{\nabla}^2\phi = -2G\nu\kappa_y x - \frac{dg}{dx} + 2G\nu\kappa_x y - \frac{df}{dy} - 2G\alpha; \quad \phi = 0 \text{ on the boundary}$$

$$x_0 W_y - y_0 W_x = \int_A (2\phi - E\kappa_x x^2 y + E\kappa_y x y^2)\, dA.$$

Some relevant formulae

$$(\boldsymbol{a} \times \boldsymbol{b})_i = \epsilon_{ijk} a_j b_k,$$

$$\sin 2\theta = 2\sin\theta\cos\theta,$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta,$$
$$\int_0^{2\pi} \cos^2\theta \sin\theta \, d\theta = 0.$$

$$\begin{aligned}
\epsilon_{rr} &= \frac{\partial u_r}{\partial r}, & \epsilon_{r\theta} &= \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right], \\
\epsilon_{\theta\theta} &= \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right), & \epsilon_{\theta z} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right), \\
\epsilon_{zz} &= \frac{\partial u_z}{\partial z}, & \epsilon_{rz} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right).
\end{aligned}$$

$$\boldsymbol{\tau} = \lambda(\operatorname{tr}\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\boldsymbol{\epsilon}.$$

$$\begin{aligned}
\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} &= 0, \\
\frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{\tau_{r\theta} + \tau_{\theta r}}{r} &= 0, \\
\frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{z\theta}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_{zr}}{r} &= 0.
\end{aligned}$$

$$[\bar{\boldsymbol{v}}] = \boldsymbol{Q}[\boldsymbol{v}]; \quad [\bar{\boldsymbol{\tau}}] = \boldsymbol{Q}[\boldsymbol{\tau}]\boldsymbol{Q}^T, \text{ where } Q_{ij} = \bar{\boldsymbol{e}}_i \cdot \boldsymbol{e}_j$$