Indian Institute of Science

ME 242: Final Exam

Date: 9/12/13.
Duration: 9.00 a.m.–12.00 noon.
Maximum Marks: 100

1. The curl of a second-order tensor $\boldsymbol{T}$, denoted by $\boldsymbol{\nabla} \times \boldsymbol{T}$, is given by (25)

$$(\boldsymbol{\nabla} \times \boldsymbol{T})_{ij} = \epsilon_{irs} \frac{\partial T_{js}}{\partial x_r}.$$

   (a) If $\boldsymbol{H} \in \text{Sym}$, find an expression for the components of $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{H})$.

   (b) Using the results in (a), find a *tensorial* expression for $\text{tr}\,[\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{H})]$ in terms of $\boldsymbol{H}$.

   (c) Let $\boldsymbol{H} \equiv \boldsymbol{\epsilon}$, where $\boldsymbol{\epsilon}$ is the strain tensor, and assume the body forces to be zero. It can be shown that $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\epsilon}) = \boldsymbol{0}$, so that $\text{tr}\,[\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\epsilon})] = 0$. Using the result in (b), the relation $\boldsymbol{\tau} = \lambda(\text{tr}\,\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\boldsymbol{\epsilon}$, the equilibrium equations, and the relation $\text{tr}\,[\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\epsilon})] = 0$, find an expression for $\boldsymbol{\nabla}^2(\text{tr}\,\boldsymbol{\tau})$.

2. Take the dot product of the equilibrium equations $\boldsymbol{\nabla}_x \cdot \boldsymbol{\tau} + \rho \boldsymbol{b} = \boldsymbol{0}$ with the position vector $\boldsymbol{x}$, and integrate over the domain V occupied by the body. Carry out an appropriate integration by parts, and find a tensorial expression in the form (25)

$$\int_V f(\boldsymbol{\tau})\, dV = \int_S \boldsymbol{t} \cdot \boldsymbol{x}\, dS + \int_V \rho \boldsymbol{b} \cdot \boldsymbol{x}, \qquad (1)$$

   where S represents the surface of V, $\boldsymbol{t}$ represents the traction vector, and $f(\boldsymbol{\tau})$ is a function of $\boldsymbol{\tau}$ that *you have to determine.*

   For the semicircular disc supported on (frictionless) roller supports, and subjected to gravity loading as shown in Fig. 1, find $\int_V f(\boldsymbol{\tau})\, dV$ using Eqn. (1). Assume the curved surface to be traction free and solve the problem as a 2D problem by neglecting effects in the out-of-plane direction. *Do not* attempt to find an exact elasticity solution to this problem.

3. The nonzero stresses within the context of the St Venant torsion theory are given by (25)

$$\tau_{xz} = G\alpha \left(\frac{\partial g}{\partial y} - y \right), \quad \tau_{yz} = G\alpha \left(-\frac{\partial g}{\partial x} + x \right). \qquad (2)$$

   (a) Using the compatibility relations

$$-\frac{\partial^2 \epsilon_{xx}}{\partial y \partial z} + \frac{\partial}{\partial x} \left(-\frac{\partial \epsilon_{yz}}{\partial x} + \frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z} \right) = 0,$$

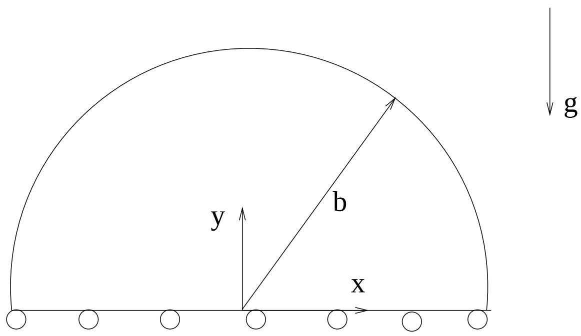

Figure 1: Semicircular disc under gravity load.

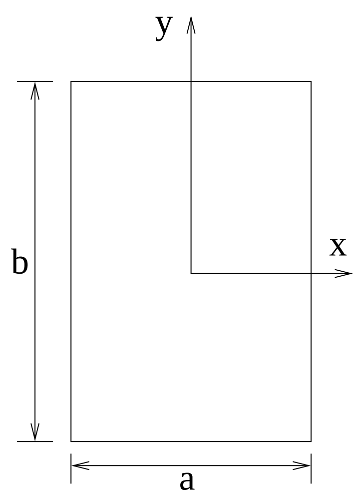

Figure 2: Rectangular bar undergoing torsion.

$$-\frac{\partial^2 \epsilon_{yy}}{\partial z \partial x} + \frac{\partial}{\partial y}\left(-\frac{\partial \epsilon_{zx}}{\partial y} + \frac{\partial \epsilon_{xy}}{\partial z} + \frac{\partial \epsilon_{yz}}{\partial x}\right) = 0,$$

and Eqns. (2), find the governing equation and boundary conditions for the conjugate function g. State any assumptions on values of constants that you choose in these equations.

(b) Consider the torsion of a beam of rectangular cross section as shown in Fig. 2 Assume the function g to be given by

$$g = c + k(y^2 - x^2) + \sum_{n=1}^{\infty} A_n \cos \alpha_n x \cosh \alpha_n y.$$

Choose α_n as a function of n such that the infinite series part vanishes on $x = \pm a/2$. Then find the constants c, k and A_n and the torsional rigidity (the expressions for A_n and the torsional rigidity can be in the form of integrals (with appropriate limits of integration) that you need not evaluate).

4. Consider the bending by terminal loads of a beam whose cross section is an (25)

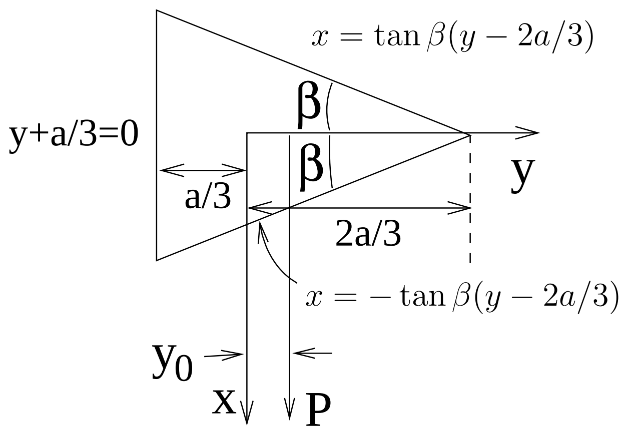

Figure 3: Bar whose cross section is an isosceles triangle acted upon by a statically equivalent load P through $y = y_0$.

isosceles triangle as shown in Fig. 3 The angle β is such that $\tan^2\beta = \nu/(1+\nu)$. Find the angle of twist α, and the stresses τ_{xz} and τ_{yz}. State the equation for determining y_0 with appropriate integration limits; *do not* attempt to evaluate this integral. You may directly use the following formulae, and guess the value of I_{xy} (but justify).

$$\kappa_x = \frac{I_{xx}W_x + I_{xy}W_y}{E(I_{xx}I_{yy} - I_{xy}^2)}, \quad \kappa_y = \frac{I_{xy}W_x + I_{yy}W_y}{E(I_{xx}I_{yy} - I_{xy}^2)}.$$

$$\tau_{xz} = \frac{\partial\phi}{\partial y} + f(y) - \frac{1}{2}E\kappa_x x^2,$$

$$\tau_{yz} = -\frac{\partial\phi}{\partial x} - g(x) - \frac{1}{2}E\kappa_y y^2,$$

$$\left[\frac{1}{2}E\kappa_x x^2 - f(y)\right]\frac{dy}{ds} = 0, \quad \left[\frac{1}{2}E\kappa_y y^2 + g(x)\right]\frac{dx}{ds} = 0,$$

$$\boldsymbol{\nabla}^2\phi = -2G\nu\kappa_y x - \frac{dg}{dx} + 2G\nu\kappa_x y - \frac{df}{dy} - 2G\alpha; \quad \phi = 0 \text{ on the boundary}$$

$$x_0W_y - y_0W_x = \int_A (2\phi - E\kappa_x x^2 y + E\kappa_y x y^2)\, dA.$$

Some relevant formulae

$$\cosh x = \frac{1}{2}(e^x + e^{-x}), \quad \frac{d(\cosh x)}{dx} = \sinh x,$$

$$\sinh x = \frac{1}{2}(e^x - e^{-x}), \quad \frac{d(\sinh x)}{dx} = \cosh x,$$

$$\boldsymbol{\tau} = \lambda(\mathrm{tr}\,\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\boldsymbol{\epsilon}, \quad \epsilon_{ijk}\epsilon_{iqr} = \delta_{jq}\delta_{kr} - \delta_{jr}\delta_{kq}.$$