Indian Institute of Science

ME 242: Final Exam

Date: 5/12/14.
Duration: 9.00 a.m.–12.00 noon.
Maximum Marks: 100

1. A solid circular bar of radius a and length L with bottom surface $z = 0$, is subjected to a radial traction on its lateral surface of the type $p_1 + p_2 z$ with the top and bottom surfaces traction free. The displacement and stresses are given by (15)

$$
\begin{aligned}
2\mu u_r &= -\left[\frac{A}{2} + (1-\nu)(\hat{A}_1 + 2\hat{A}_2 z)\right] r, \\
2\mu u_z &= (A + 2\nu\hat{A}_1)z + \left[(1-\nu)r^2 + 2\nu z^2\right]\hat{A}_2, \\
\tau_{rr} &= -\frac{A}{2} - (1+\nu)(\hat{A}_1 + 2\hat{A}_2 z), \\
\tau_{\theta\theta} &= -\frac{A}{2} - (1+\nu)(\hat{A}_1 + 2\hat{A}_2 z), \\
\tau_{zz} &= A, \\
\tau_{rz} &= 0.
\end{aligned}
$$

Assuming that the stress field satisfies the equilibrium equations, evaluate the constants A, $\hat{A}_1$ and $\hat{A}_2$.

2. Take the dot product of the equilibrium equations $\boldsymbol{\nabla}_x \cdot \boldsymbol{\tau} + \rho\boldsymbol{b} = \boldsymbol{0}$ with the displacement vector $\boldsymbol{u}$, and integrate over the domain V occupied by the body. By using appropriate tensorial identities (that you should derive), find a tensorial expression that relates the strain energy to the work done by the loads. (25)

Now consider a circular cylinder of radius a and length L with polar moment of inertia J that is subjected to a torque T by applying suitable tractions on the top surface $z = L$ with the lateral surfaces traction free, and the bottom surface $z = 0$ fixed. By taking

$$
\begin{aligned}
u_\theta &= \alpha r z, \\
\tau_{\theta z} &= \frac{Tr}{J},
\end{aligned}
$$

with the remaining displacement and stress components zero, evaluate the constant α using the equation that you have derived earlier. Ignore body forces.

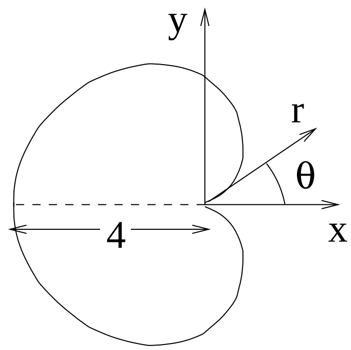

Figure 1: Problem 3.

3. A prismatic bar with cross section as shown in Fig. 1 is subjected to a distribution of tractions on its top and bottom surfaces with the lateral surfaces traction free. The equation of the surface of the cross section is given by $r = 4\sin^2(\theta/2)$ (or $\sqrt{r} = 2\sin(\theta/2)$). The expressions for the nonzero stresses (which satisfy the equilibrium equations) are (30)

$$\begin{aligned}
\tau_{rz} &= c_1 r^{-1/2}\cos\frac{\theta}{2} + c_2\sin\theta,\\
\tau_{\theta z} &= c_3 r + c_2\cos\theta - c_1 r^{-1/2}\sin\frac{\theta}{2},\\
\tau_{zz} &= c_4 r\sin\theta.
\end{aligned}$$

By using the appropriate boundary conditions, find the constants c_1, c_2, c_3 and c_4 in terms of the torque T and bending moment M applied about the x-axis. Note that a stress singularity at $r = 0$ is permitted in this problem (due to the notch there). Also write the expressions for the net force vector on the top cross section. *You need not evaluate the integrals that arise, but they should have the proper integration limits.*

4. Consider the bending by a terminal load of a beam of rectangular cross section as shown in Fig. 2 The nonzero stresses in terms of a harmonic function χ are given by (30)

$$\begin{aligned}
\frac{\tau_{xz}}{G} &= -\alpha y + \frac{\partial\chi}{\partial x} - \frac{\kappa_x}{4}\left[3x^2 + y^2 + 2\nu(x^2 - y^2)\right] - \frac{\kappa_y xy}{2}(1+2\nu),\\
\frac{\tau_{yz}}{G} &= \alpha x + \frac{\partial\chi}{\partial y} - \frac{\kappa_x xy}{2}(1+2\nu) - \frac{\kappa_y}{4}\left[x^2 + 3y^2 + 2\nu(y^2 - x^2)\right],\\
\tau_{zz} &= -E(L-z)(\kappa_x x + \kappa_y y),\\
\kappa_x &= \frac{I_{xx}W_x + I_{xy}W_y}{E(I_{xx}I_{yy} - I_{xy}^2)}, \quad \kappa_y = \frac{I_{xy}W_x + I_{yy}W_y}{E(I_{xx}I_{yy} - I_{xy}^2)}.
\end{aligned}$$

You may guess the value of α (with proper justification). *Choose* the appropriate expression for χ from among

$$\chi = c_1(x^3 - 3xy^2) + c_2 x + \frac{1}{G}\sum_{m=1}^{\infty}\frac{A_m a}{(2m-1)\pi}\sin\frac{(2m-1)\pi x}{a}\cosh\frac{(2m-1)\pi y}{a},$$

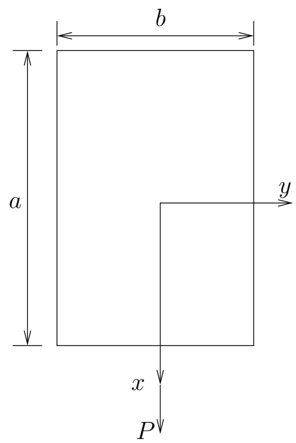

Figure 2: Bar of rectangular cross-section acted upon by a statically equivalent load P through its centroid.

OR

$$\chi = c_1(x^3 - 3xy^2) + c_2 x + \frac{1}{G}\sum_{m=1}^{\infty} \frac{A_m a}{2m\pi} \cos\frac{2m\pi x}{a} \sinh\frac{2m\pi y}{a}.$$

based on symmetry considerations (Justify your choice). Then solve for the constants c_1, c_2 and A_m using the appropriate boundary conditions. Integrals in the denominator should be evaluated. *You need not evaluate the integrals that arise in the numerator, but they should have the proper integration limits.*

Some relevant formulae

$$\boldsymbol{\nabla}\phi = \begin{bmatrix} \frac{\partial \phi}{\partial r} \\ \frac{1}{r}\frac{\partial \phi}{\partial \theta} \\ \frac{\partial \phi}{\partial z} \end{bmatrix},$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x}), \quad \frac{d(\cosh x)}{dx} = \sinh x,$$

$$\sinh x = \frac{1}{2}(e^x - e^{-x}), \quad \frac{d(\sinh x)}{dx} = \cosh x,$$

$$\boldsymbol{\tau} = \lambda(\operatorname{tr}\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\boldsymbol{\epsilon}, \quad \sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}, \quad \cos\theta = 1 - 2\sin^2\frac{\theta}{2}.$$