Indian Institute of Science

ME 242: Final Exam

Date: 29/11/19.
Duration: 9.30 a.m.–12.30 p.m.
Maximum Marks: 100

1. Let $\boldsymbol{a}$, $\boldsymbol{b}$ be two arbitrary vectors. (25)

   (a) Determine if $\boldsymbol{a} \otimes \boldsymbol{b} - \boldsymbol{b} \otimes \boldsymbol{a} \in$ Skw. If it is, find using indicial notation or otherwise, the axial vector of $\boldsymbol{a} \otimes \boldsymbol{b} - \boldsymbol{b} \otimes \boldsymbol{a}$ in terms of $\boldsymbol{a} \times \boldsymbol{b}$.

   (b) A body in equilibrium occupies a volume V with surface S. By integrating the equations

   $$(\boldsymbol{\nabla} \cdot \boldsymbol{\tau}) \otimes \boldsymbol{x} + \rho \boldsymbol{b} \otimes \boldsymbol{x} = \boldsymbol{0},$$

   over the domain V, and with the use of the divergence theorem, find a relation between $\int_V \boldsymbol{\tau}\, dV$ on the left hand side, and $\int_S \boldsymbol{t} \otimes \boldsymbol{x}\, dS$ and $\int_V \rho \boldsymbol{b} \otimes \boldsymbol{x}\, dV$, on the right hand side. Take the transpose of this relation. Since $\boldsymbol{\tau} \in$ Sym, the left hand side remains the same, but the right hand side expression is now different. Show that the two right hand side expressions that you have obtained are equal. *Justify* all steps.

   (c) A circular cylinder, free of tractions on all its surfaces, is subjected to a body force $\rho \boldsymbol{b} = c_0 r \boldsymbol{e}_r$, where c_0 is a constant. Using the above results, find $\int_V \tau_{zz}\, dV$.

2. A wedge of inner radius a and outer radius b, and of unit width, is subjected to a bending moment M under plane stress conditions as shown in Fig. 1. The surfaces $\theta = \pm\theta_0$ are free of tractions. On the curved surfaces, the net resultant forces along the x and y-axis are zero. By assuming (30)

   $$\tau_{rr} = \frac{-2(2c_2 \sin 2\theta + c_3 r \sin\theta)}{r^2},$$
   $$\tau_{\theta\theta} = 0,$$
   $$\tau_{r\theta} = \frac{c_1 + 2c_2 \cos 2\theta}{r^2},$$

   find the constants c_1, c_2 and c_3 in terms of M. You may directly use the fact that the integral of a function odd in θ such as $\sin\theta$ or $\sin\theta\cos\theta$ over the domain $[-\theta_0, \theta_0]$ is zero. For the remaining integrals, use the formulae at the end.

3. The solution for the torsion of a cracked circular cylinder with the crack (25)

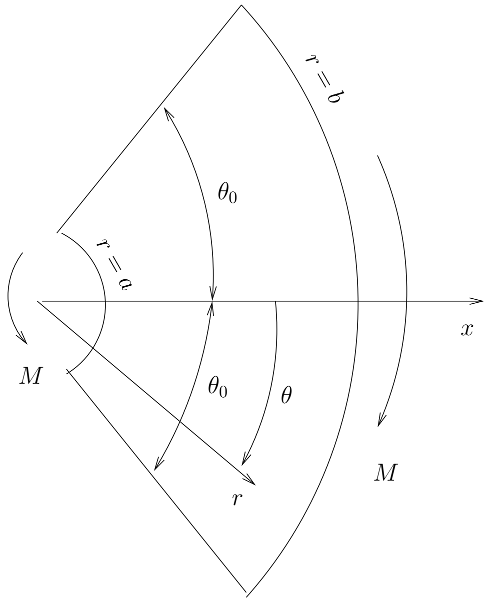

Figure 1: Wedge subjected to bending moment on its curved surfaces.

extending radially outward from the center of the cross section to the periphery (with the crack extending along the entire length of the cylinder) can be found by considering the torsion of a wedge similar to that shown in Fig. 1 but with $a = 0$ and with $\theta_0 = \pi$. Thus, let the Prandtl stress function ϕ which satisfies $\boldsymbol{\nabla}^2\phi = -2G\alpha$, be given by

$$\frac{\phi}{G\alpha} = -\frac{r^2}{2} + c_0 r^2 \cos\alpha\theta - \sum_{m=1}^{\infty} A_m r^{\beta_m} \cos\gamma_m\theta, \tag{1}$$

where $\beta_m > 0$ for all m, and the first term on the right hand side $\tilde{\phi} = -G\alpha r^2/2$ satisfies $\boldsymbol{\nabla}^2\tilde{\phi} = -2G\alpha$.

(a) Using the fact that harmonic functions are the real and imaginary parts of a complex-valued function $W(z)$, where $z = re^{i\theta}$, find α, and the relation between β_m and γ_m.

(b) Using the expressions for the stress components,

$$\tau_{rz} = \frac{1}{r}\frac{\partial\phi}{\partial\theta},$$
$$\tau_{\theta z} = -\frac{\partial\phi}{\partial r},$$

and the fact that all the lateral surfaces of the wedge (cracked cylinder) are traction free, find all the unknown constants in Eqn. (1) (Do *not* use $\phi = 0$ on the boundary).

(c) Is there a stress singularity at $r = 0$; if a singularity exists, and if the term causing the singularity is of the form $1/r^g$, then what is the value of g?

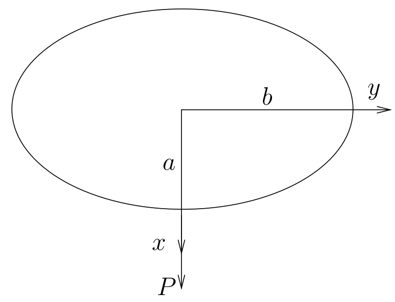

Figure 2: Bar with elliptical cross-section acted upon by a terminal load.

(d) Using the fact that

$$g = \frac{1}{2}(x^2 + y^2) + \frac{\phi}{G\alpha},$$

and the fact that ψ and g are conjugate harmonic functions, find an expression for ψ (in this expression, you need not substitute the values of the constants which you have found in the earlier parts).

4. Consider the bending by a terminal load of a beam of elliptical cross section as shown in Fig. 2. The nonzero stresses in terms of a harmonic function χ are given by (20)

$$\begin{aligned}
\frac{\tau_{xz}}{G} &= -\alpha y + \frac{\partial \chi}{\partial x} - \frac{\kappa_x}{4}\left[3x^2 + y^2 + 2\nu(x^2 - y^2)\right] - \frac{\kappa_y xy}{2}(1 + 2\nu), \\
\frac{\tau_{yz}}{G} &= \alpha x + \frac{\partial \chi}{\partial y} - \frac{\kappa_x xy}{2}(1 + 2\nu) - \frac{\kappa_y}{4}\left[x^2 + 3y^2 + 2\nu(y^2 - x^2)\right], \\
\tau_{zz} &= -E(L - z)(\kappa_x x + \kappa_y y), \\
\kappa_x &= \frac{I_{xx}W_x + I_{xy}W_y}{E(I_{xx}I_{yy} - I_{xy}^2)}, \quad \kappa_y = \frac{I_{xy}W_x + I_{yy}W_y}{E(I_{xx}I_{yy} - I_{xy}^2)}.
\end{aligned}$$

You may guess the value of α (with proper justification). The expression for χ is given by

$$\chi = c_1 x + c_2 x^3 + c_3 xy^2.$$

(a) If $\phi(x, y)$ is the equation of a curve, then find an expression for the unit normal $\boldsymbol{n}$ to this curve in terms of $\partial\phi/\partial x$ and $\partial\phi/\partial y$ by first stating the expression for the tangent in terms of dx/ds and dy/ds (s is a parameter that parametrizes the curve), and then using the fact that the normal and tangent are perpendicular.

(b) Find κ_x and κ_y. State the value of I_{xy} (with justification), and evaluate I_{yy} using the transformation $x = ar\cos\theta$, $y = br\sin\theta$.

(c) State the equations for finding the constants c_1, c_2 and c_3 by using the fact that χ is harmonic, and using the boundary conditions at the two points $(a, 0)$ and $(a/\sqrt{2}, b/\sqrt{2})$ on the ellipse. *Do not* attempt to either simplify or solve these equations.

Some relevant formulae

$$w_i = -\frac{1}{2}\epsilon_{ijk}W_{jk},$$
$$W_{ij} = -\epsilon_{ijk}w_k,$$

$$\int_{-\theta_0}^{\theta_0} \cos\theta \, d\theta = 2\sin\theta_0,$$
$$\int_{-\theta_0}^{\theta_0} \cos 2\theta \, d\theta = \sin 2\theta_0,$$
$$\int_{-\theta_0}^{\theta_0} \cos^2\theta \, d\theta = \theta_0 + \frac{\sin 2\theta_0}{2},$$
$$\int_{-\theta_0}^{\theta_0} \sin^2\theta \, d\theta = \theta_0 - \frac{\sin 2\theta_0}{2},$$
$$\int_{-\theta_0}^{\theta_0} \cos\theta\cos 2\theta \, d\theta = (1+\cos^2\theta_0)\sin\theta_0 - \frac{\sin^3\theta_0}{3},$$
$$\int_{-\theta_0}^{\theta_0} \sin\theta\sin 2\theta \, d\theta = \frac{4\sin^3\theta_0}{3},$$
$$\int_{-\pi}^{\pi} \sin 2\theta\sin\gamma_m\theta \, d\theta = \frac{4\sin\gamma_m\pi}{\gamma_m^2-4},$$
$$\int_{-\pi}^{\pi} \cos 2\theta\cos\gamma_m\theta \, d\theta = \frac{2\gamma_m\sin\gamma_m\pi}{\gamma_m^2-4}.$$