Indian Institute of Science

ME 242: Final Exam

Date: 22/1/21.
Duration: 9.15 a.m.–1.15 p.m.
Maximum Marks: 100

1. Let $\boldsymbol{W} \in$ Skw and $\boldsymbol{Q} \in$ Orth$^+$. (15)

   (a) Determine whether $\boldsymbol{Q}\boldsymbol{W}^2\boldsymbol{Q}^T$ is symmetric, skew-symmetric, orthogonal or none of the above.

   (b) Determine the eigenvalue/eigenvectors of $\boldsymbol{Q}\boldsymbol{W}^2\boldsymbol{Q}^T$. If the eigenvectors are nonunique, *choose* a set of eigenvectors, and present your choice.

   (c) It is possible for tensors to fall into two categories. For example, the diagonal matrix **diag**$[1,-1,-1]$ is both symmetric and orthogonal. Determine if $\boldsymbol{Q}\boldsymbol{W}^2\boldsymbol{Q}^T$ can be orthogonal. If it can be, give an example, if it cannot be, present a mathematical justification.

2. A circular hole of radius a in an unbounded domain is subjected to a traction distribution $\boldsymbol{t} = (p_0 + p_1\cos\theta)\boldsymbol{e}_r$ (with (r,θ) denoting polar coordinates) on its surface. The Airy stress function and the corresponding stresses are given by (15)

$$
\begin{aligned}
\phi &= F_2\theta + \frac{A_2a^3(\nu-1)\cos\theta}{2r} + \frac{B_2a^3(1-\nu)\sin\theta}{2r} + C_2r^3\cos\theta + a^2F_1\log r \\
&\quad + r\cos\theta\left[2H_2\theta + J_2a(\nu-1)\log r\right] + r\sin\theta\left[2J_2a\theta + aH_2(1-\nu)\log r\right], \\
\tau_{rr} &= \frac{F_1a^2}{r^2} + 2C_2r\cos\theta + \frac{A_2a^3(1-\nu)\cos\theta}{r^3} + \frac{J_2a(1+\nu)\cos\theta}{r} \\
&\quad - \frac{B_2a^3(1-\nu)\sin\theta}{r^3} - \frac{H_2(1+\nu)a\sin\theta}{r}, \\
\tau_{\theta\theta} &= -\frac{F_1a^2}{r^2} + 6C_2r\cos\theta - \frac{A_2a^3(1-\nu)\cos\theta}{r^3} + \frac{J_2a(\nu-1)\cos\theta}{r} \\
&\quad + \frac{B_2a^3(1-\nu)\sin\theta}{r^3} + \frac{H_2(1-\nu)a\sin\theta}{r}, \\
\tau_{r\theta} &= \frac{F_2}{r^2} + 2C_2r\sin\theta + \frac{A_2a^3(1-\nu)\sin\theta}{r^3} - \frac{J_2a(1-\nu)\sin\theta}{r} \\
&\quad + \frac{B_2a^3(1-\nu)\cos\theta}{r^3} - \frac{H_2a(1-\nu)\cos\theta}{r}.
\end{aligned}
$$

   Determine the various constants (total seven in number) in the above formulation.

3. Consider the torsion of a bar whose cross section is a triangle with angles (30)

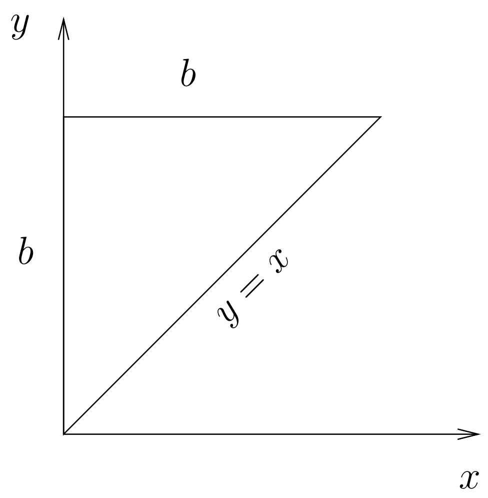

Figure 1: Torsion of a bar whose cross-section is a triangle with angles 45°-45°-90°.

45°-45°-90° as shown in Fig. 1. The Prandtl stress function is given by

$$\frac{\phi}{G\alpha} = c_0 x(y-x) - \sum_{m=1}^{\infty} A_m \left[\sin(\beta_m x)\sinh(\beta_m y) - \sin(\beta_m y)\sinh(\beta_m x)\right],$$

(a) Determine the constants c_0, A_m and β_m (where the constants with subscript m are dependent on m). Some of these constants can be in the form of an integral with appropriate limits; you need not evaluate these integrals.

(b) Write the expression for the torsional rigidity as an integral with appropriate limits. *Do not* evaluate this integral.

4. A cracked beam of circular cross section of radius b with the crack extending from the center to the periphery, and along the entire length $[0, L]$ is loaded by a statically equivalent load of P acting at the centroid and directed along the y-axis as shown in Fig. 2. (40)

(a) State with justification (but without mathematical proof), the value of twist at the centroid α.

(b) State the value of I_{xy} (without proof), and use it to find the 'curvatures'

$$\kappa_x = \frac{I_{xx}W_x + I_{xy}W_y}{E(I_{xx}I_{yy} - I_{xy}^2)}; \quad \kappa_y = \frac{I_{yy}W_y + I_{xy}W_x}{E(I_{xx}I_{yy} - I_{xy}^2)},$$

in terms of I_{xx} and I_{yy} (which you need not evaluate).

(c) Use the formulae

$$\left[\frac{1}{2}E\kappa_x x^2 - f(y)\right]\frac{dy}{ds} - \left[\frac{1}{2}E\kappa_y y^2 + g(x)\right]\frac{dx}{ds} = 0,$$

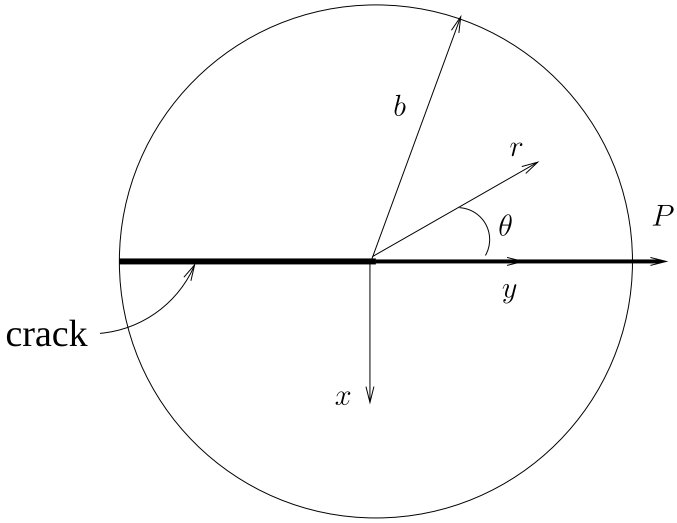

Figure 2: Problem 4.

$$\nabla^2\phi = -2G\nu\kappa_y x - \frac{dg}{dx} + 2G\nu\kappa_x y - \frac{df}{dy} - 2G\alpha,$$

to determine ϕ.

(d) Find the relations for τ_{rz} and $\tau_{\theta z}$ in terms of τ_{xz}, τ_{yz} and (r, θ) (*do not* substitute for τ_{xz} and τ_{yz} in these expressions.)

Some relevant formulae

$$\begin{aligned} w_i &= -\frac{1}{2}\epsilon_{ijk}W_{jk}, \\ W_{ij} &= -\epsilon_{ijk}w_k, \\ \boldsymbol{W} &= |\boldsymbol{w}|\,(\boldsymbol{r}\otimes\boldsymbol{q} - \boldsymbol{q}\otimes\boldsymbol{r}),\ \{\boldsymbol{w}/|\boldsymbol{w}|\,, \boldsymbol{q}, \boldsymbol{r}\} \text{ orthonormal} \end{aligned}$$