Indian Institute of Science

ME 242: Final Exam

Date: 03/12/21.
Duration: 9.15 a.m.–1.15 p.m.
Maximum Marks: 100

1. This problem deals with the uniqueness of the field variables in linear elasticity. (25)

   (a) Starting from the equilibrium equation $\boldsymbol{\nabla} \cdot \boldsymbol{\tau} + \rho \boldsymbol{b} = \boldsymbol{0}$, and assuming a linear elastic material with constitutive relation $\boldsymbol{\tau} = \mathbf{C}\boldsymbol{\epsilon}$ where $\mathbf{C}$ is symmetric and positive definite, derive the Clayperon equation that relates the work done by the loads to the strain energy.

   (b) For the two-dimensional plane-strain problem shown in Fig. 1, state the in-plane boundary conditions at $r = a$ and at $\theta = 0$ and $\theta = \pi$.

   (c) In class, we considered the uniqueness of solution to the linear elasticity problem under pure traction, pure displacement, and 'mixed' boundary value problems where traction were prescribed on part of the boundary and displacements on the remaining part. The problem shown in Fig. 1 does not fall in any of these categories since the traction is prescribed along one direction, and the displacement prescribed along a direction orthogonal to it *on the same boundary.* For the problem shown in Fig. 1, modify the arguments made in class, and determine if $\boldsymbol{u}$, $\boldsymbol{\epsilon}$ and $\boldsymbol{\tau}$ are unique (*without explicitly finding the solution*).

   (d) If any or all of these three fields are nonunique, give the exact nature of the nonuniqueness for the problem shown in Fig. 1 for the inplane variables only (e.g., if you find that $\boldsymbol{\epsilon}$ is nonunique, then present an expression for the inplane $\boldsymbol{\epsilon}^{(1)} - \boldsymbol{\epsilon}^{(2)}$).

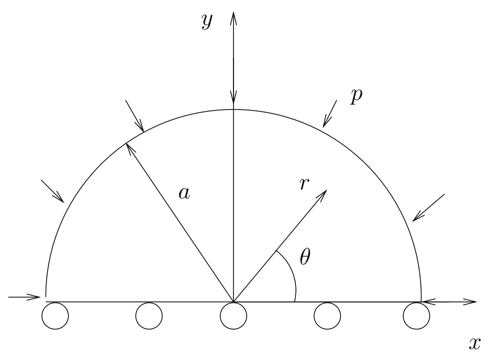

Figure 1: A semicircular domain supported by roller supports on its flat edge, and subjected to uniform pressure loading on its curved edge.

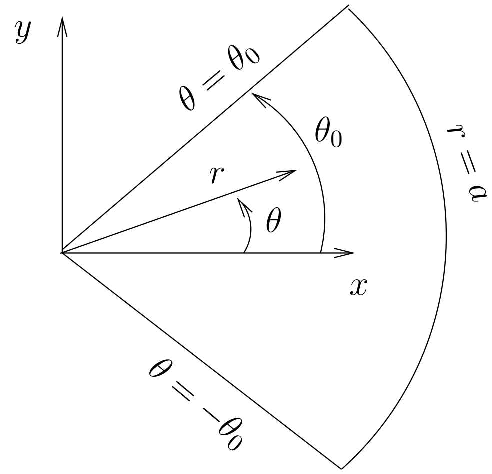

Figure 2: Torsion of a bar whose cross-section is a sector of a circle.

2. Consider the torsion of a bar whose cross section is a sector of a circle as shown in Fig. 2. (30)

   (a) The boundary condition for the warping function is

   $$(\boldsymbol{\nabla}\psi) \cdot \boldsymbol{n} = y n_x - x n_y.$$

   Write this boundary condition in the polar coordinate system shown in the figure for the edges $r = a$ and $\theta = \pm\theta_0$ with $\boldsymbol{n} = (n_r, n_\theta)$.

   (b) The warping function that automatically satisfies the governing equation is given by

   $$\psi = -\frac{r^2 \sin 2\theta}{2\cos 2\theta_0} + \sum_{m=1}^{\infty} \Big\{ \left(A_m r^{\gamma_m} - B_m r^{-\gamma_m}\right) \sin \gamma_m \theta + \left(C_m r^{\gamma_m} + D_m r^{-\gamma_m}\right) \cos \gamma_m \theta \Big\},$$

   where γ_m is a function of m and θ_0. Determine the constants A_m, B_m, C_m, D_m and γ_m. You may make an intelligent choice for γ_m based on the boundary conditions; the remaining constants have to be determined in a systematic manner from the boundary and symmetry conditions. You need not carry out any integrations that arise in the expressions for these constants, but the integrals should be stated with the proper integrand and integration limits. *Justify* all steps.

3. A prismatic bar of triangular cross section is restricted on all surfaces except the top one by a rigid frictionless container (see Fig. 3 where the top and front views are shown). The material is isotropic and linear elastic with Young modulus and Poisson ratio E and ν, respectively. The top surface is subjected to a uniform pressure loading of p. We assume the stress field to be given by $\tau_{xx} = -p_1$, $\tau_{yy} = -p_2$ and $\tau_{zz} = -p$, with all the remaining (25)

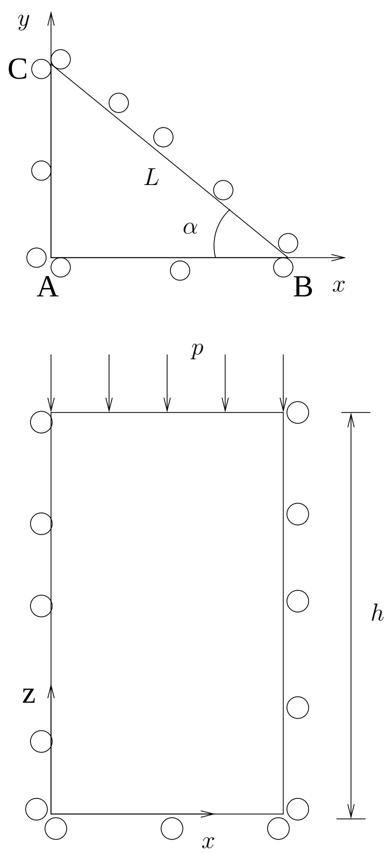

Figure 3: Top and front views of a triangular cross section bar in a rigid frictionless container subjected to a uniform pressure p on its top surface.

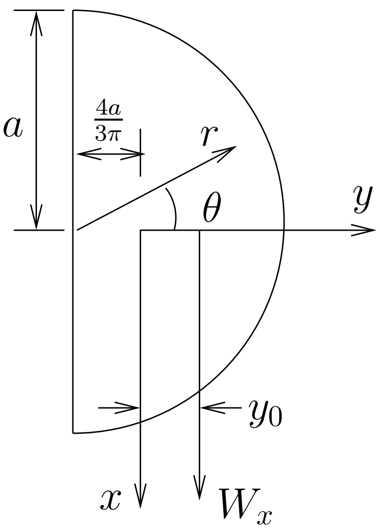

Figure 4: Bar of semi-circular cross-section subjected to bending by a terminal load W_x.

stress components zero (where p_1 and p_2 are unknowns to be determined as mentioned below).

(a) State the in-plane boundary conditions on the surfaces AB, BC and AC in the plane x-y. These boundary conditions should be stated in terms of the in-plane displacement and nonzero stress components.

(b) Using the strain-stress relations (i.e., strain components expressed in terms of stresses) for an isotropic linear elastic material, find the strain field.

(c) Using the normal strain-displacement relations, find the displacement field (i.e., do not consider the shear strain components). Take the integration functions/constants while carrying out the integrations to be zero. For example,

$$u_x = \int \epsilon_{xx}\, dx.$$

(d) Using the in-plane boundary conditions that you have stated earlier, determine p_1 and p_2 in terms of p.

4. A semicircular beam is subjected to a loading by a terminal load W_x as shown in Fig. 4. The stress distribution is given by (20)

$$\begin{aligned}\frac{\tau_{rz}}{G} &= \frac{C_0(3+2\nu)(r^2-k_1^2)\sin\theta}{4}, \\ \frac{\tau_{\theta z}}{G} &= \frac{C_0\left[(1-2\nu)r^2-(3+2\nu)a^2\right](\cos\theta+k_2\sin\theta)}{4}.\end{aligned}$$

Find the constants C_0, k_1 and k_2. The value of y_0 is not required for solving the stated problem.

Some relevant formulae

$$\int_{-\theta_0}^{\theta_0} \cos\theta \, d\theta = 2\sin\theta_0,$$

$$\int_{-\theta_0}^{\theta_0} \cos 2\theta \, d\theta = \sin 2\theta_0,$$

$$\int_{-\theta_0}^{\theta_0} \sin 2\theta \, d\theta = 0,$$

$$\int_{-\theta_0}^{\theta_0} \cos^2\theta \, d\theta = \theta_0 + \frac{\sin 2\theta_0}{2},$$

$$\int_{-\theta_0}^{\theta_0} \sin^2\theta \, d\theta = \theta_0 - \frac{\sin 2\theta_0}{2},$$

$$\int_{-\theta_0}^{\theta_0} \sin\frac{m\pi\theta}{\theta_0} \sin\frac{n\pi\theta}{\theta_0} \, d\theta = \delta_{mn}\theta_0,$$

$$\int_{-\theta_0}^{\theta_0} \cos\frac{m\pi\theta}{\theta_0} \cos\frac{n\pi\theta}{\theta_0} \, d\theta = \delta_{mn}\theta_0,$$

$$\int_{-\theta_0}^{\theta_0} \sin\frac{(2m-1)\pi\theta}{2\theta_0} \sin\frac{(2n-1)\pi\theta}{2\theta_0} \, d\theta = \delta_{mn}\theta_0,$$

$$\int_{-\theta_0}^{\theta_0} \cos\frac{(2m-1)\pi\theta}{2\theta_0} \cos\frac{(2n-1)\pi\theta}{2\theta_0} \, d\theta = \delta_{mn}\theta_0.$$