Indian Institute of Science

ME 242: Final Exam

Date: 1/12/22.
Duration: 9.00 a.m.–12.00 noon.
Maximum Marks: 100

1. Let S denote the surface, V denote the domain of a body, $\boldsymbol{t}$ denote the static traction field acting on S, and $\boldsymbol{\tau}$ the stress field. The body force and the acceleration can be assumed to be zero throughout this problem. (30)

   (a) If $\boldsymbol{Q}(t) \in \text{Orth}^+$, determine if $\boldsymbol{Q}^T\dot{\boldsymbol{Q}}$ is skew-symmetric, symmetric, orthogonal or none of these.

   (b) If
   $$\int_S \boldsymbol{Q}\boldsymbol{t} \cdot (\dot{\boldsymbol{Q}}\boldsymbol{x})\, dS = \alpha,$$
   where α is a constant, determine α by converting the left hand side of the above equation to a volume integral using the divergence theorem, and then simplifying. *Justify* every step in your derivation. You may directly use the Cauchy relation relating the traction and stress fields.

2. A rigid inclusion of radius a is inserted into a hole of radius a in an unbounded domain, and its surface is bonded to the surface of the hole as shown in Fig. 1. The rigid inclusion is rotated by an angle ϕ about its axis so that for example, the point A on the periphery of the hole is now displaced to point B as shown in the figure. The relevant plane-stress solution is given by (25)

$$\begin{aligned}
2Eu_r &= -\frac{2(1+\nu)F_1}{r} + 4(1-\nu)r\hat{C}_1 + 2\left[(1-3\nu)r^2\hat{C}_2 + \frac{(1+\nu)A_{-2}}{r^2}\right]\cos\theta \\
&\quad + 2\left[(1-3\nu)r^2\hat{D}_2 + \frac{(1+\nu)B_{-2}}{r^2}\right]\sin\theta \\
&\quad + (1+\nu)(3-\nu)\log r\,(J_2\cos\theta - H_2\sin\theta) \\
2Eu_\theta &= -\frac{2(1+\nu)F_2}{r} + 2\left[(5+\nu)r^2\hat{C}_2 + \frac{(1+\nu)A_{-2}}{r^2}\right]\sin\theta \\
&\quad - 2\left[(5+\nu)r^2\hat{D}_2 + \frac{(1+\nu)B_{-2}}{r^2}\right]\cos\theta \\
&\quad - H_2(1+\nu)(1+\nu+(3-\nu)\log r)\cos\theta \\
&\quad - J_2(1+\nu)(1+\nu+(3-\nu)\log r)\sin\theta \\
\tau_{rr} &= \frac{F_1}{r^2} + 2\hat{C}_1 + 2\left(r\hat{C}_2 - \frac{A_{-2}}{r^3}\right)\cos\theta + 2\left(r\hat{D}_2 - \frac{B_{-2}}{r^3}\right)\sin\theta \\
&\quad + \frac{(3+\nu)\,(J_2\cos\theta - H_2\sin\theta)}{2r}
\end{aligned}$$

$$
\begin{aligned}
\tau_{\theta\theta} &= -\frac{F_1}{r^2} + 2\hat{C}_1 + 2\left(3r\hat{C}_2 + \frac{A_{-2}}{r^3}\right)\cos\theta + 2\left(3r\hat{D}_2 + \frac{B_{-2}}{r^3}\right)\sin\theta \\
&+ \frac{(1-\nu)\,(H_2\sin\theta - J_2\cos\theta)}{2r} \\
\tau_{r\theta} &= \frac{F_2}{r^2} + 2\left(r\hat{C}_2 - \frac{A_{-2}}{r^3}\right)\sin\theta - 2\left(r\hat{D}_2 - \frac{B_{-2}}{r^3}\right)\cos\theta \\
&- \frac{(1-\nu)\,(H_2\cos\theta + J_2\sin\theta)}{2r}.
\end{aligned}
$$

State the boundary conditions with respect to the polar coordinate system (Hint: Find in Cartesian and then transform to polar). Determine the nonzero constants in the above solution. Finally, determine the in-plane principal stresses at $r = a$ in terms of the nonzero constants (you need not simplify by substituting for these constants).

3. Consider the torsion of a beam of rectangular cross section as shown in Fig. 2 From among the various choices (30)

$$\psi = xy + \sum_{n=1}^{\infty} A_n \cos\alpha_n x \cosh\alpha_n y, \tag{1a}$$

$$\psi = xy + \sum_{n=1}^{\infty} A_n \sin\alpha_n x \cosh\alpha_n y, \tag{1b}$$

$$\psi = xy + \sum_{n=1}^{\infty} A_n \cos\alpha_n x \sinh\alpha_n y, \tag{1c}$$

$$\psi = xy + \sum_{n=1}^{\infty} A_n \sin\alpha_n x \sinh\alpha_n y, \tag{1d}$$

choose the correct form of the warping function by providing a *justification* for your choice. Next, determine the constants α_n and A_n as a function of n using the boundary condition

$$(\boldsymbol{\nabla}\psi)\cdot\boldsymbol{n} = yn_x - xn_y.$$

In your results, you can have one and only one unevaluated integral (which should be stated with the proper integration limits though).

4. Consider the bending by a terminal load $W_y = P$ along the y-axis of a beam of semi-parabolic cross section as shown in Fig. 3. With $\kappa_y = P/(EI_{xx})$, the nonzero stresses are given by (15)

$$
\begin{aligned}
\frac{\tau_{xz}}{\kappa_y G} &= -\alpha y + \frac{x(c_1 - 3y)}{6}, \\
\frac{\tau_{yz}}{\kappa_y G} &= \alpha x + \frac{(c_2 - 3y)(y + a)}{3}.
\end{aligned}
$$

You may guess the value of α (with proper justification). Then solve for the constants c_1 and c_2. Let A denote the domain occupied by the cross section of the beam at $z = c$ ($z = 0$ and $z = L$ are the left and right ends of the beam). *Without* actually evaluating any integrals state the values of

$$\int_A \tau_{xz}\, dA = ? \qquad \int_A x\tau_{zz}\, dA = ?$$
$$\int_A \tau_{yz}\, dA = ? \qquad \int_A y\tau_{zz}\, dA = ?$$
$$\int_A \tau_{zz}\, dA = ? \qquad \int_A (x\tau_{yz} - y\tau_{xz})\, dA = ?$$

Some relevant formulae

$$\cosh x = \frac{1}{2}(e^x + e^{-x}), \qquad \frac{d(\cosh x)}{dx} = \sinh x,$$
$$\sinh x = \frac{1}{2}(e^x - e^{-x}), \qquad \frac{d(\sinh x)}{dx} = \cosh x,$$
$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi,$$
$$\sin(\theta + \phi) = \sin\theta\cos\phi + \sin\phi\cos\theta.$$

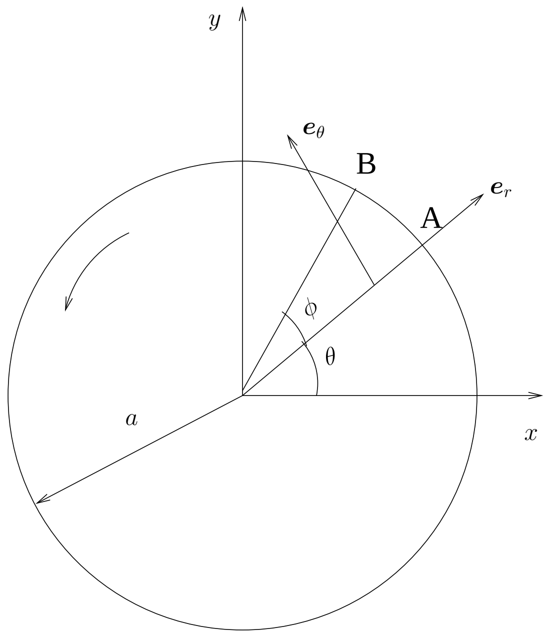

Figure 1: Perfectly bonded rigid inclusion in a circular hole of radius a rotated by an angle ϕ about its axis.

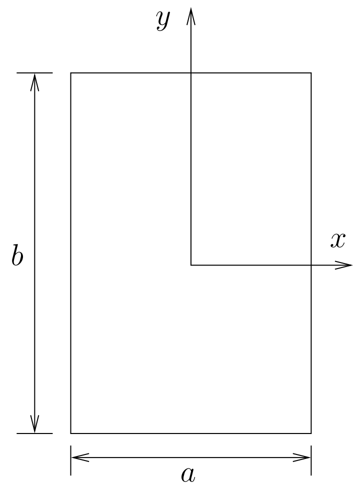

Figure 2: Rectangular bar undergoing torsion.

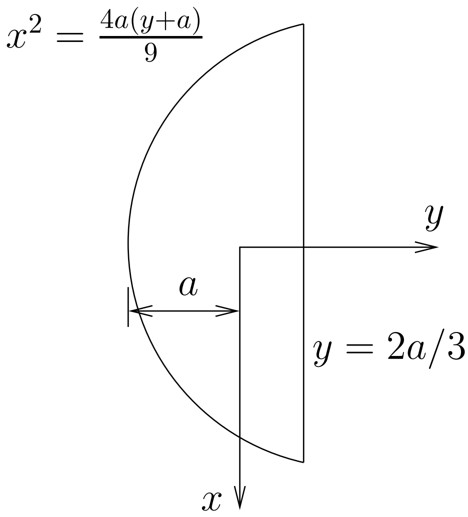

Figure 3: Bar of semi-parabolic cross-section subjected to a terminal load along the y-direction.