Indian Institute of Science

ME 242: Final Exam

Date: 24/12/25.
Duration: 9.00 a.m.–12.00 noon.
Maximum Marks: 100

1. This is an exercise in tensor analysis, and no knowledge of transient solutions is required. $\boldsymbol{a}$ and $\boldsymbol{p}$ are constant vectors, c is a constant, and $\boldsymbol{x}$ and t denote the position vector and time. (20)

   (a) Substitute the solution
   $$\boldsymbol{u}(\boldsymbol{x}, t) = \boldsymbol{a} \sin(\boldsymbol{p} \cdot \boldsymbol{x} - ct), \quad |\boldsymbol{p}| = 1,$$
   into the Navier equations of elasticity
   $$\rho \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = (\lambda + \mu) \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + \mu \boldsymbol{\nabla}^2 \boldsymbol{u},$$
   and simplify each term.

   (b) After canceling common terms, write the resulting equation in the form
   $$\boldsymbol{A}(\boldsymbol{p})\boldsymbol{a} = c^2 \boldsymbol{a},$$
   where $\boldsymbol{A}$ is a second order tensor that you have to determine.

   (c) Is $\boldsymbol{A}$ symmetric? Determine the eigenvalues/eigenvectors of $\boldsymbol{A}$. If the eigenvectors are not unique, then state one set of orthonormal eigenvectors.

2. A circular hole of radius a in an unbounded domain is subjected to a far-field stress $\tau_{xy} = s_0$, i.e., $\lim_{r \to \infty} \tau_{xy} = s_0$ under plane strain conditions. It is given that the Airy stress function is of the form $\phi = f(r) \sin(m\theta)$. (25)

   (a) Using the far-field loading as a guide, guess the value of m.

   (b) Using the fact that the real and imaginary parts of $z^n = (re^{i\theta})^n$ are harmonic functions, and the fact that if β is harmonic, then β and $r^2\beta$ are biharmonic, write an expression for $f(r)$ in terms of undetermined constants (e.g., $f(r) = c_1 r^4 + c_2 r^6$).

   (c) State the appropriate boundary conditions for this problem, and using these boundary conditions, and the expressions
   $$\tau_{rr} = \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} + \frac{1}{r} \frac{\partial \phi}{\partial r},$$
   $$\tau_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2},$$
   $$\tau_{r\theta} = -\frac{\partial^2}{\partial r \partial \theta} \left(\frac{\phi}{r} \right).$$

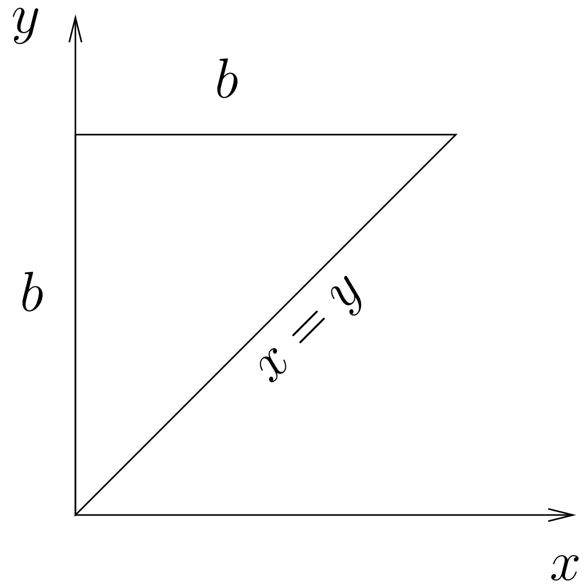

Figure 1: A bar with a right-angled triangular cross section undergoing torsion.

determine the constants in your expression for $f(r)$.

3. Consider the torsion of a beam whose cross section is a right-angled triangle as shown in Fig. 1 With $\lambda_n = n\pi/b$, the warping function which satisfies the governing equation $\boldsymbol{\nabla}^2\psi = 0$ is given by (25)

$$\psi = c_1 xy + c_2(x^2 - y^2) + \sum_{n=1}^{\infty} \frac{A_n\left[\cos(\lambda_n x)\cosh(\lambda_n y) + \cosh(\lambda_n x)\cos(\lambda_n y)\right]}{\lambda_n \sinh(\lambda_n b)}.$$

The boundary condition is given by

$$(\boldsymbol{\nabla}\psi)\cdot\boldsymbol{n} = yn_x - xn_y.$$

Determine the various constants. Using the fact that $D \leq GJ$, find an upper bound on the torsional rigidity. *Do not* evaluate any integrals in the expressions for the constants, or the upper bound on the torsional rigidity. However, these integrals must be stated with proper integration limits.

4. A cracked beam of circular cross section of radius b with the crack extending from the center to the periphery, and along the entire length $[0, L]$ is loaded by a statically equivalent load of P acting at the shear center (*do not* try to determine the shear center) and directed parallel to the x-axis as shown in Fig. 2. If $(\bar{x}_c, \bar{y}_c)$ denote the coordinates of the centroid from the origin of the cylindrical system (with θ measured anticlockwise from the y-axis), the solution in the cylindrical coordinate system, obtained using the relations $x + \bar{x}_c = -r\sin\theta$, $y + \bar{y}_c = r\cos\theta$, are (30)

$$\kappa_x = \frac{I_{xx}W_x + I_{xy}W_y}{E(I_{xx}I_{yy} - I_{xy}^2)},$$

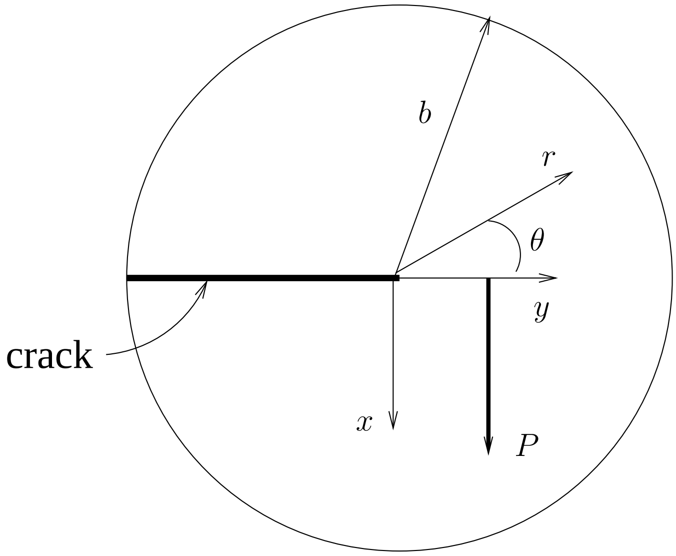

Figure 2: Problem 4.

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\begin{aligned}
\kappa_y &= \frac{I_{xy}W_x + I_{yy}W_y}{E(I_{xx}I_{yy} - I_{xy}^2)}, \\
\chi &= -\frac{\alpha r^2 \sin 2\theta}{2} + c_1 r \sin\theta + c_2 r^2 \sin 2\theta + c_3 r^3 \sin 3\theta + \sum_{n=1}^{\infty} \frac{A_n r^{\lambda_n} \sin(\lambda_n \theta)}{\lambda_n b^{\lambda_n - 1}}, \\
\frac{\tau_{rz}}{G} &= \alpha\left(-\bar{x}_c \cos\theta - \bar{y}_c \sin\theta\right) + \frac{\partial\chi}{\partial r} \\
&+ \frac{\kappa_x}{4}\Big\{4(1+\nu) r \bar{x}_c + \left[(3+2\nu)(r^2 + \bar{x}_c^2) + (1-2\nu)\bar{y}_c^2\right]\sin\theta \\
&- 2r\bar{x}_c \cos 2\theta - 2(1+2\nu)\bar{x}_c\bar{y}_c \cos\theta - 2r\bar{y}_c \sin 2\theta\Big\} \\
&+ \frac{\kappa_y}{4}\Big\{(4 + 4\nu + 2\cos 2\theta) r \bar{y}_c + 2(1+2\nu)\bar{x}_c\bar{y}_c \sin\theta - 2r\bar{x}_c \sin 2\theta \\
&- \left[(3+2\nu)(r^2 + \bar{y}_c^2) + (1-2\nu)\bar{x}_c^2\right]\cos\theta\Big\}, \\
\frac{\tau_{\theta z}}{G} &= \alpha\left(r + \bar{x}_c \sin\theta - \bar{y}_c \cos\theta\right) + \frac{1}{r}\frac{\partial\chi}{\partial\theta} \\
&+ \frac{\kappa_x}{4}\Big\{2r\bar{y}_c(2\nu - \cos 2\theta) + \left[(1-2\nu)(r^2 + \bar{y}_c^2) + (3+2\nu)\bar{x}_c^2\right]\cos\theta \\
&+ 2\bar{x}_c\bar{y}_c(1+2\nu)\sin\theta + 2r\bar{x}_c \sin 2\theta\Big\} \\
&+ \frac{\kappa_y}{4}\Big\{-2r\bar{x}_c(\cos 2\theta + 2\nu) + \left[(1-2\nu)(r^2 + \bar{x}_c^2) + (3+2\nu)\bar{y}_c^2\right]\sin\theta \\
&+ 2(1+2\nu)\bar{x}_c\bar{y}_c \cos\theta - 2r\bar{y}_c \sin 2\theta\Big\}, \\
\tau_{zz} &= E(L - z)\left[\kappa_x(r\sin\theta + \bar{x}_c) - \kappa_y(r\cos\theta - \bar{y}_c)\right].
\end{aligned}
$$

(a) State the value of α (with proper justification).

(b) Find $\bar{x}_c$, $\bar{y}_c$, I_{xx}, I_{yy} and I_{xy} for this cross section.

(c) Choose λ_n such that the *infinite series part* of the boundary condition on the crack faces is zero (note that the stress can be singular at the crack tip).

(d) Determine c_1, c_2, c_3 and A_n, $n = 1, 2, \ldots, \infty$. *Do not* evaluate any integrals in the expressions for the constants. However, these integrals must be stated with proper integration limits.

(e) At the cross section $z = z_0$, *without* actually evaluating any integrals (but with justification) state the values of

$$\int_A \tau_{xz}\, dA = ? \qquad \int_A x\tau_{zz}\, dA = ?$$

$$\int_A \tau_{yz}\, dA = ? \qquad \int_A y\tau_{zz}\, dA = ?$$

$$\int_A \tau_{zz}\, dA = ?$$

Some relevant formulae

$$Q_{ij} = \bar{\boldsymbol{e}}_i \cdot \boldsymbol{e}_j,$$

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta,$$

$$\sin(2\theta) = 2\sin\theta\cos\theta,$$

$$\cosh x = \frac{1}{2}(e^x + e^{-x}), \qquad \frac{d(\cosh x)}{dx} = \sinh x,$$

$$\sinh x = \frac{1}{2}(e^x - e^{-x}), \qquad \frac{d(\sinh x)}{dx} = \cosh x,$$

$$\boldsymbol{\mathcal{I}} = \int_A \left[(\boldsymbol{x}\cdot\boldsymbol{x})\boldsymbol{I} - \boldsymbol{x}\otimes\boldsymbol{x}\right] dA.$$