Indian Institute of Science

ME 242: Final Exam

Date: 4/12/03.
Duration: 2.30 p.m.–5.30 p.m.
Maximum Marks: 100

1. Calculate the stresses at the **center** of a compound disc spinning at 1000 rpm as shown in Fig. 1. (25)

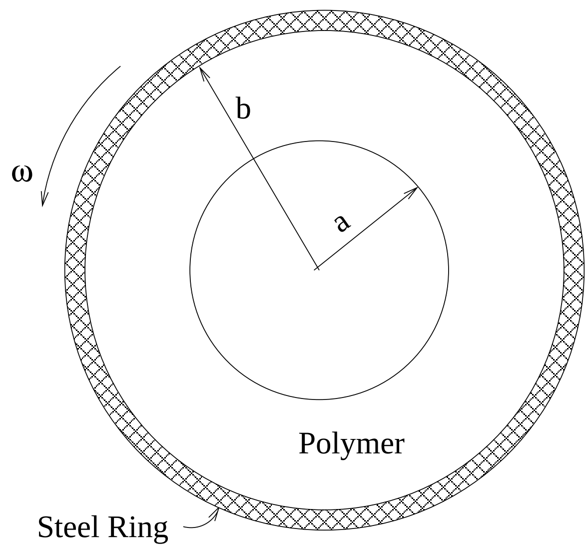

Figure 1: Spinning compound disc.

2. The curl of a second-order tensor $\boldsymbol{T}$, denoted by $\boldsymbol{\nabla} \times \boldsymbol{T}$, is given by (20)

$$(\boldsymbol{\nabla} \times \boldsymbol{T})_{ij} = \epsilon_{irs} \frac{\partial T_{js}}{\partial x_r}.$$

   (a) If $\boldsymbol{S} \in \text{Sym}$, find an expression for the components of $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{S})$, and using this expression and the symmetry of $\boldsymbol{S}$, determine if $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{S})$ is symmetric.

   (b) Show that $\boldsymbol{\nabla} \cdot [\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{S})] = \boldsymbol{0}$.

   (c) Using the relation $\boldsymbol{\epsilon} = 0.5[\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^t]$ for the strain tensor and the relation for $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{S})$ derived in part (a), show that $\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{\epsilon}) = \boldsymbol{0}$. (Note: The six independent components of this tensor constitute the *compatibility conditions*).

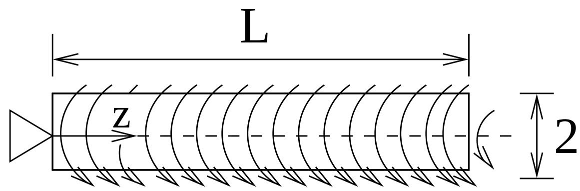

Figure 2: Torque applied on the lateral *and* end surfaces.

3. In the class, we considered the problem of a circular cylinder subjected to a torque applied through tractions acting at the end surfaces, with the lateral surfaces being free of traction. In this problem, we consider a circular cylinder subjected to a torque through a traction distribution on the lateral and end surface ($z = L$) as shown in Fig. 2. The bar is fixed at $(r, z) = (0, 0)$ as shown, and we assume that $u_r = u_z = 0$ along the entire bar, i.e., all particles only move in tangential directions. Based on this assumption, it follows that $\tau_{r\theta}$ and $\tau_{\theta z}$ are the only nonzero components of stress and if they are given by (30)

$$\tau_{r\theta} = -\frac{1}{r^2}\frac{\partial \phi}{\partial z}; \quad \tau_{\theta z} = \frac{1}{r^2}\frac{\partial \phi}{\partial r},$$

then the equations of equilibrium are automatically satisfied.

(a) Using the relations

$$\begin{aligned} \tau_{r\theta} &= Gr\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) = -\frac{1}{r^2}\frac{\partial \phi}{\partial z}, \\ \tau_{\theta z} &= Gr\frac{\partial}{\partial z}\left(\frac{u_\theta}{r}\right) = \frac{1}{r^2}\frac{\partial \phi}{\partial r}, \end{aligned} \tag{1}$$

eliminate u_θ (by carrying out appropriate partial differentiations) and find out the governing equation for ϕ. (Hint: This governing equation will involve terms such as $\partial^2\phi/\partial r^2$ etc.)

(b) Verify whether or not the function $\phi = czr^4$, where c is a given constant, satisfies this governing differential equation.

(c) Determine the traction distributions on the lateral and end surfaces corresponding to the above ϕ, and find the *total* resultant torque exerted about the z-axis due to these distributions.

(d) Using Eqns. (1), and the given boundary conditions, find an expression for u_θ.

4. A beam with an equilateral cross section is loaded by a statically equivalent load of P acting at the centroid and directed along the x-axis as shown in Fig. 3. Since the shear center is at the centroid, no twisting of the cross-section occurs ($\alpha = 0$). (25)

(a) State the value of I_{xy} (without proof), and use it to find the 'curvatures'

$$\kappa_x = \frac{I_{xx}W_x + I_{xy}W_y}{E(I_{xx}I_{yy} - I_{xy}^2)}; \quad \kappa_y = \frac{I_{yy}W_y + I_{xy}W_x}{E(I_{xx}I_{yy} - I_{xy}^2)},$$

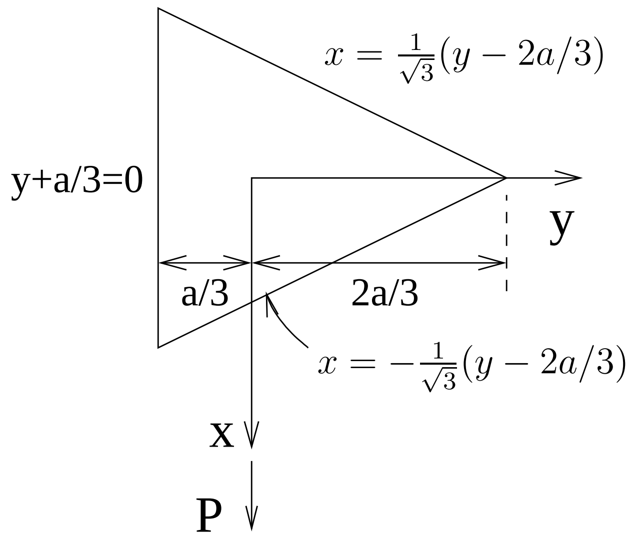

Figure 3: Beam of triangular cross section loaded by a statically equivalent load of P through its centroid.

in terms of I_{xx} and I_{yy} (which you need not evaluate).

(b) Find the functions $g(x)$ and $f(y)$ such that the boundary condition

$$\left[\frac{1}{2}E\kappa_x x^2 - f(y)\right]\frac{dy}{ds} - \left[\frac{1}{2}E\kappa_y y^2 + g(x)\right]\frac{dx}{ds} = 0$$

is satisfied. (Hint: See if $f(y)$ can be determined by considering part of the surface).

(c) Given that $\nu = 0.5$, use the governing differential equation

$$\boldsymbol{\nabla}^2\phi = -2G\nu\kappa_y x - \frac{\partial g}{\partial x} + 2G\nu\kappa_x y - \frac{\partial f}{\partial y},$$

to find ϕ (Hint: $\phi = 0$ on the boundary). Note that once ϕ is known, the stresses can be determined, and the problem is solved.