Indian Institute of Science

ME 242: Final Exam

Date: 3/12/09.
Duration: 9.30 a.m.–12.30 noon
Maximum Marks: 100

1. A sandwich compound *solid* disk of radius $b = 1$ m (see Fig. 1) is spinning with an angular velocity of $\omega = 300$ rad/sec. The inner material A and the outer material B have the same Poisson ratio $\nu_A = \nu_B = 1/3$. The Young moduli and densities of the two materials are $E_A = 200$ GPa and (20)

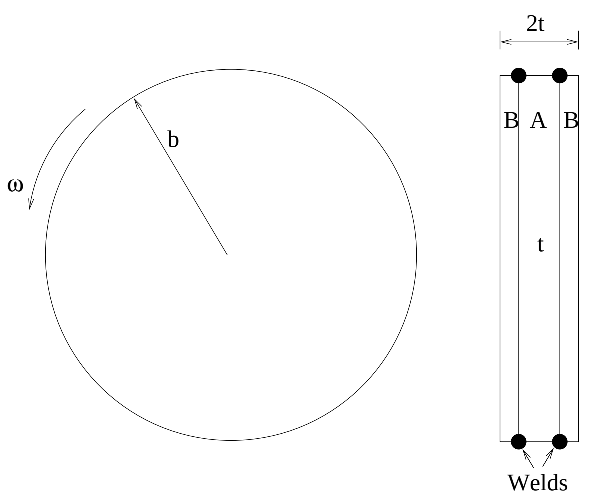

Figure 1: Compound spinning disc.

$E_B = 100$ GPa, $\rho_A = 8000$ kg/m^3, $\rho_B = 2000$ kg/m^3. Tabulate the radial and tangential stresses at the center ($r = 0$) and at the rim ($r = 1$ m) in both materials *with* and *without* the weld. Assume plane stress conditions to prevail independently in the discs.

r (m)	Material	σ_r (MPa) no weld	σ_θ (MPa) no weld	σ_r (MPa) with weld	σ_θ (MPa) with weld
0	A	?	?	?	?
1	A	?	?	?	?
0	B	?	?	?	?
1	B	?	?	?	?

2. We have seen that the solution to the pure bending problem by a moment for a beam of length L and of circular cross section (see Fig. 2) is given by (20)

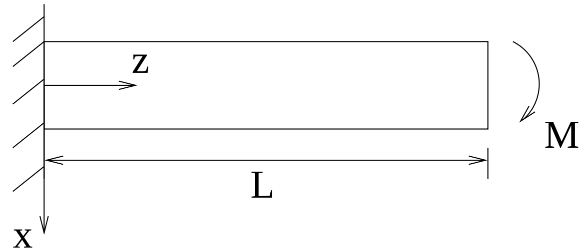

Figure 2: A prismatic beam of circular cross section subjected to a bending moment.

$$\tau_{zz} = -\frac{Mx}{I_{yy}},$$

with other stress components zero, while the displacement field, obtained using the boundary conditions $u_x = u_y = u_z = 0$ and $\frac{\partial u_x}{\partial z} = \frac{\partial u_y}{\partial z} = \frac{\partial u_y}{\partial x} = 0$ at $(x, y, z) = (0, 0, 0)$, is given by

$$\begin{aligned} u_x &= \frac{M}{2EI_{yy}}\left[z^2 + \nu(x^2 - y^2)\right], \\ u_y &= \frac{\nu xyM}{EI_{yy}}, \\ u_z &= -\frac{xzM}{EI_{yy}}. \end{aligned}$$

(a) Using the strain displacement relation, find the strain field, and compute the strain energy W in the entire beam (you can express your answer in terms of I_{yy} without evaluating I_{yy}).

(b) Using the strength of materials relation

$$\theta = \frac{\partial W}{\partial M},$$

find the angle θ at the tip $z = L$ in the x-z plane. Also find the angle (by making suitable approximations such as $\tan\theta \approx \theta$) using the displacement field given above. Do the two results match?

3. In order to solve the torsion problem for a beam with a semi-circular cross-section (see Fig. 3), we assume the Prandtl stress function to be of the form (25)

$$\phi = cr^2(1 + \cos 2\theta) + G\alpha \sum_{m=1}^{\infty} A_m r^{2m-1} \cos(2m-1)\theta.$$

Using the relation

$$\boldsymbol{\nabla}^2\phi \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = -2G\alpha,$$

and the boundary condition $\phi = 0$ on the boundary, find the constants c and A_m, $m = 1, 2, \ldots, \infty$. You may directly use the fact that the real and imaginary parts of z^n (n integer and $z = re^{i\theta}$) are harmonic.

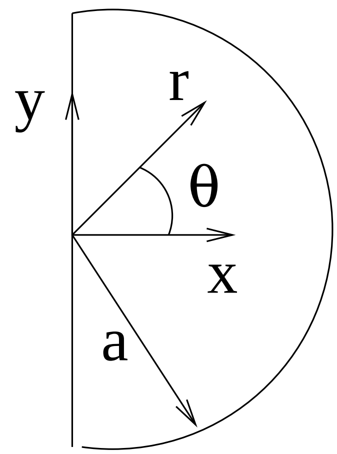

Figure 3: Torsion of a semicircular bar.

4. The problem of bending by terminal loads can be solved in an alternative manner as follows. The displacements are assumed to be (35)

$$
\begin{aligned}
u_x &= -\alpha yz + \kappa_x \left[\frac{1}{2}\nu(L-z)(x^2-y^2) - \frac{1}{6}z^3 + \frac{1}{2}Lz^2\right] + \kappa_y \nu (L-z)xy, \\
u_y &= \alpha xz + \kappa_x \nu (L-z)xy + \kappa_y \left[\frac{1}{2}\nu(L-z)(y^2-x^2) - \frac{1}{6}z^3 + \frac{1}{2}Lz^2\right], \\
u_z &= \left(\frac{1}{2}z^2 - Lz\right)(\kappa_x x + \kappa_y y) - \chi(x,y) - \kappa_x x y^2 - \kappa_y x^2 y,
\end{aligned}
$$

where α, as usual, represents the twist per unit length at the centroid, and $\chi(x,y)$ is a function to be determined. The corresponding stresses (with $G = E/(2(1+\nu))$ representing the shear modulus) are

$$\frac{\tau_{xz}}{G} = -\alpha y - \frac{\partial \chi}{\partial x} - \kappa_x \left[y^2 + \frac{\nu}{2}(x^2-y^2)\right] - \kappa_y xy(2+\nu), \tag{1a}$$

$$\frac{\tau_{yz}}{G} = \alpha x - \frac{\partial \chi}{\partial y} - \kappa_x xy(2+\nu) - \kappa_y \left[x^2 + \frac{\nu}{2}(y^2-x^2)\right], \tag{1b}$$

$$\tau_{zz} = -E(L-z)(\kappa_x x + \kappa_y y), \tag{1c}$$

with the other components of stresses zero.

(a) Using the equations of equilibrium, find the governing equation for χ.

(b) We want to solve the problem shown in Fig. 4. *State* (without proving) the value of I_{xy} and α, and using the relations

$$\kappa_x = \frac{I_{xx}W_x + I_{xy}W_y}{E(I_{xx}I_{yy} - I_{xy}^2)}, \quad \kappa_y = \frac{I_{xy}W_x + I_{yy}W_y}{E(I_{xx}I_{yy} - I_{xy}^2)},$$

find κ_x and κ_y (you can express your answers in terms of I_{xx}, I_{yy}, I_{xy}), and substitute in Eqns. (1) to get expressions for τ_{xz}, τ_{yz} and τ_{zz}.

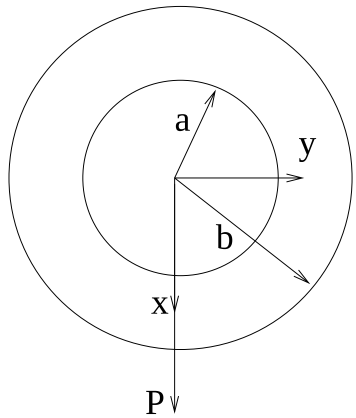

Figure 4: Bending of a hollow circular cylinder by a terminal load P.

(c) It is convenient to work with cylindrical coordinates. Hence, transform the stress matrix $\begin{bmatrix} 0 & 0 & \tau_{xz} \\ 0 & 0 & \tau_{yz} \\ \tau_{xz} & \tau_{yz} & \tau_{zz} \end{bmatrix}$ to cylindrical polar coordinates. Using the chain rule and the relations $x = r\cos\theta$, $y = r\sin\theta$, find expressions for the stress components in the cylindrical coordinate system in terms of (r, θ) (the derivatives $\partial\chi/\partial x$ and $\partial\chi/\partial y$ should also be transformed to $\partial\chi/\partial r$ and $\partial\chi/(r\partial\theta)$).

(d) Assume $\chi = c_1 r\cos\theta + c_2\cos\theta/r + c_3 r^3\cos 3\theta$. Does this form satisfy the governing equation for χ that you have derived? (again you may directly use the fact that the real and imaginary parts of z^n (n integer and $z = re^{i\theta}$) are harmonic)

(e) Find the constants c_1, c_2, c_3 using the appropriate boundary conditions on the stresses.

Some relevant formulae

$$\int_{-\pi/2}^{\pi/2} \cos(2m-1)\theta\cos(2p-1)\theta\, d\theta = \begin{cases} 0 & (\text{if } m \neq p) \\ \pi/2 & (\text{if } m = p) \end{cases},$$

$$\int_{-\pi/2}^{\pi/2} (1+\cos 2\theta)\cos(2m-1)\theta\, d\theta = \frac{8(-1)^m}{(4m^2-1)(2m-3)}.$$

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta, \\ \sin 2\theta &= 2\sin\theta\cos\theta, \\ \cos 3\theta &= 4\cos^3\theta - 3\cos\theta, \\ \sin 3\theta &= 3\sin\theta - 4\sin^3\theta. \end{aligned}$$

$$[\bar{\boldsymbol{\tau}}] = \boldsymbol{Q}[\boldsymbol{\tau}]\boldsymbol{Q}^T, \text{ where } Q_{ij} = \bar{\boldsymbol{e}}_i \cdot \boldsymbol{e}_j$$