# Indian Institute of Science

# ME 242: Final Exam

**Date:** 6/12/10.
**Duration:** 9.30 a.m.–12.30 noon
**Maximum Marks:** 100

1. A circular cylinder of radius $a$ is subjected to a uniform temperature change of $T_\Delta$ under plane strain conditions ($u_z = \epsilon_{zz} = 0$) as shown in Fig. 1a. Assuming the problem to be axisymmetric, i.e., $u_r = u_r(r)$, the Navier equations reduce to (20)

$$\frac{d}{dr}\left[\frac{1}{r}\frac{d(ru_r)}{dr}\right] = 0.$$

   (a) Find a general solution to the above equation.

   (b) Using the relations $\epsilon_{rr} = \partial u_r/\partial r$, $\epsilon_{\theta\theta} = u_r/r$ (other strain components zero), $\boldsymbol{\tau} = \lambda(\operatorname{tr}\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\boldsymbol{\epsilon} - (3\lambda + 2\mu)\alpha T_\Delta \boldsymbol{I}$, and appropriate boundary conditions, find the expressions for the displacements and stresses.

   (c) Now consider a composite cylinder with the inner and outer cylinders of radii $a$ and $b$, and with material properties $(\lambda_1, \mu_1, \alpha_1)$ and $(\lambda_2, \mu_2, \alpha_2)$, respectively. Based on the boundary conditions, write the appropriate equations that one would need to solve in order to find a solution to this problem. *Do not attempt* to solve these equations.

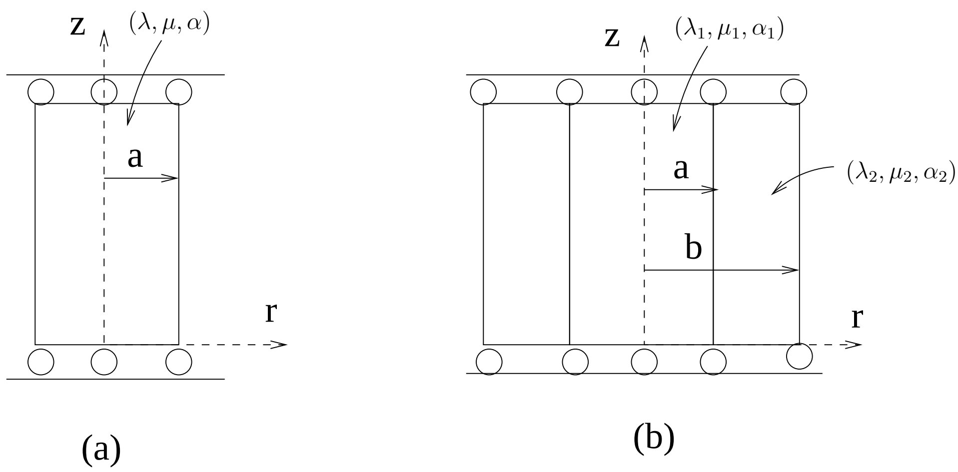


Figure 1: Circular cylinder subjected to a uniform temperature change $T_\Delta$: (a) Single material (b) Composite cylinder.

2. Our goal in this problem is to write the Navier equations of elasticity given (20)

by

$$\rho\frac{\partial^2 \boldsymbol{u}}{\partial t^2} = (\lambda+\mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{u}) + \mu\boldsymbol{\nabla}^2\boldsymbol{u} + \rho\boldsymbol{b}$$

in terms of stresses. The material properties $(\lambda, \mu, \rho)$ may be treated as constants. Note that although you can use indicial notation to derive your result, the final equation must be in tensorial form.

(a) Write $\boldsymbol{\nabla}\cdot\boldsymbol{u}$ in the above equation in terms of $\boldsymbol{\epsilon}$.

(b) Take the '$\boldsymbol{\nabla}$' (gradient) of the resulting equation.

(c) Take the transpose of the equation obtained in (b), and add it to the equation in (b) to get an equation in terms of strains.

(d) Invert the relation $\boldsymbol{\tau} = \lambda(\operatorname{tr}\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\boldsymbol{\epsilon}$ to find $\boldsymbol{\epsilon}$ as a function of $\boldsymbol{\tau}$.

(e) Substitute the relation in (d) into the equation obtained in (c) to get the desired result.

3. The torsion of circular shafts of variable diameter by tractions on the end faces can be treated by assuming that $u_\theta$ is the only nonzero displacement component. This leads to $\tau_{r\theta}$ and $\tau_{\theta z}$ being the only nonzero components of stress. If they are expressed in terms of a function $\phi$ as (35)

$$\tau_{r\theta} = -\frac{1}{r^2}\frac{\partial\phi}{\partial z}, \quad \tau_{\theta z} = \frac{1}{r^2}\frac{\partial\phi}{\partial r}, \tag{1}$$

then the equations of equilibrium are automatically satisfied.

(a) Using the relations

$$\begin{aligned} \tau_{r\theta} &= Gr\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right) = -\frac{1}{r^2}\frac{\partial\phi}{\partial z}, \\ \tau_{\theta z} &= Gr\frac{\partial}{\partial z}\left(\frac{u_\theta}{r}\right) = \frac{1}{r^2}\frac{\partial\phi}{\partial r}, \end{aligned} \tag{2}$$

eliminate $u_\theta$, and find the (second-order) governing partial differential equation for $\phi$.

(b) Verify if the function $\phi = czr^4$, where $c$ is a given constant, satisfies this governing differential equation.

(c) Consider the cylinder shown in Fig. 2 whose boundary is defined by the equation $zr^4 = k$, where $k$ is a constant, Taking $\phi = czr^4$, find the stresses using Eqn. (1), and determine if the lateral sides of this cylinder are traction free. Also find $u_\theta$ using Eqn. (2) (assume that rigid displacements have been suppressed).

(d) Find the total torque acting on the top and bottom faces $z = z_2$ and $z = z_1$ (whose outer radii are given by $(k/z_2)^{1/4}$ and $(k/z_1)^{1/4}$, respectively). Are the two torques equal? Also find the total force on the top surface.

4. A beam with a cross section in the shape of an isosceles triangle is loaded by a statically equivalent load of $P$ directed along the $y$-axis as shown in Fig. 3. The material is incompressible (i.e., $G = E/3$). (25)

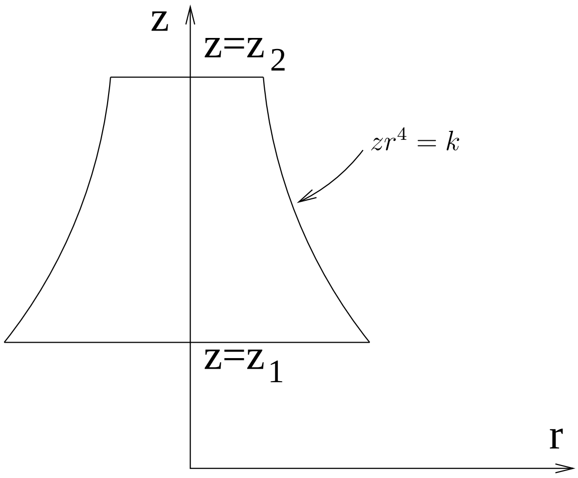


Figure 2: Circular cylinder of variable diameter subjected to torsion.

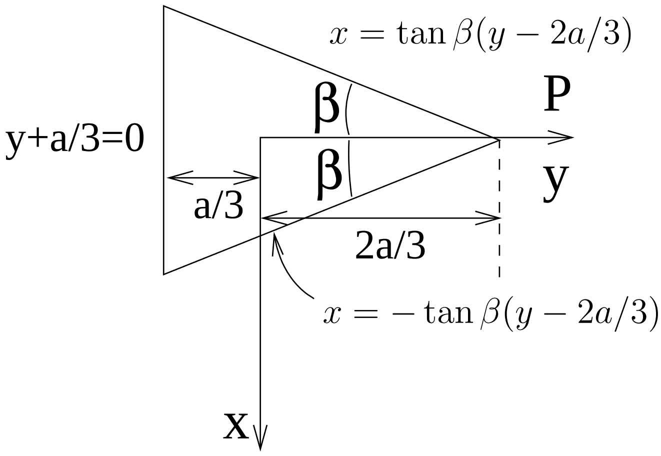


Figure 3: Beam of triangular cross section loaded by a statically equivalent load $P$ along the $y$-axis.

(a) State the value of $\alpha$ and $I_{xy}$ (without proof), and use it to find the 'curvatures'

$$\kappa_x = \frac{I_{xx}W_x + I_{xy}W_y}{E(I_{xx}I_{yy} - I_{xy}^2)}; \quad \kappa_y = \frac{I_{yy}W_y + I_{xy}W_x}{E(I_{xx}I_{yy} - I_{xy}^2)},$$

in terms of $I_{xx}$ and $I_{yy}$ (which you need not evaluate).

(b) The stress distribution is given by

$$\tau_{xz} = -G\kappa_y x\left(y + \frac{a}{3}\right),$$
$$\tau_{yz} = G\kappa_y\left(y + \frac{a}{3}\right)(c_1 + c_2 y).$$

Using the boundary conditions, determine $c_1$ and $c_2$.

(c) State (without proof, but with justification), the value of $\int_A \tau_{yz}\, dA$ and $\int_A (x\tau_{yz} - y\tau_{xz})\, dA$.