Indian Institute of Science
ME 242: Midsemester Test

Date: 7/10/12.
Duration: 2.30 p.m.–4.00 p.m.
Maximum Marks: 100

1. Let $\{I_1, I_2, I_3\}$ and $\{J_1, J_2, J_3\}$ denote the principal invariants of a symmetric tensor $\boldsymbol{S}$ and its inverse $\boldsymbol{S}^{-1}$, respectively. Find expressions for $\{J_1, J_2, J_3\}$ in terms of $\{I_1, I_2, I_3\}$. (20)

2. If $\boldsymbol{w}$ is the axial vector of skew-symmetric tensor $\boldsymbol{W}$, then we have $\boldsymbol{W}\boldsymbol{u} = \boldsymbol{w} \times \boldsymbol{u}$ for any vector $\boldsymbol{u}$. (25)

   (a) Using the above relation, find an expression for $\boldsymbol{W}_1\boldsymbol{W}_2$ in terms of the axial vectors $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ of the skew-symmetric tensors $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$. *Derive* any expressions that you need on the way.

   (b) Find an expression for $\boldsymbol{W}_1\boldsymbol{W}_2 + \boldsymbol{W}_2\boldsymbol{W}_1$ in terms of $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$. Is this tensor symmetric?

   (c) Assuming $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ to be linearly independent, find *one* eigenvalue/eigenvector pair for $\boldsymbol{W}_1\boldsymbol{W}_2 + \boldsymbol{W}_2\boldsymbol{W}_1$.

3. If $\boldsymbol{a}$ and $\boldsymbol{b}$ are vector fields, using indicial notation, find a relation between $\boldsymbol{\nabla}\cdot[\boldsymbol{a}\times(\boldsymbol{\nabla}\times\boldsymbol{b})]$, $(\boldsymbol{\nabla}\times\boldsymbol{a})\cdot(\boldsymbol{\nabla}\times\boldsymbol{b})$ and $\boldsymbol{a}\cdot[\boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{b})]$. Integrate this relation that you find over a closed domain V with surface S, use the divergence theorem, and write the resulting relationship in tensorial form. (35)

4. The stress field in an isotropic sphere of radius R and density ρ expressed in the spherical coordinate system (r, θ, ϕ) is given by (20)

$$\tau_{rr} = k(5\lambda + 6\mu)\left[\left(\frac{r}{R}\right)^2 - 1\right],$$

$$\tau_{\theta\theta} = \tau_{\phi\phi} = k(5\lambda + 2\mu)\left[\left(\frac{r}{R}\right)^2 - \frac{5\lambda + 6\mu}{5\lambda + 2\mu}\right],$$

with the other stress components zero. (λ, μ, k) are all constant. Find the body force vector field acting on the sphere, and the tractions acting at the surface of the sphere.

Some relevant formulae

$$(\boldsymbol{\nabla}\cdot\boldsymbol{\tau})_r = \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{r\phi}}{\partial \phi} + \frac{1}{r}\left(2\tau_{rr} - \tau_{\theta\theta} - \tau_{\phi\phi} + \cot\theta\,\tau_{r\theta}\right),$$

$$(\boldsymbol{\nabla}\cdot\boldsymbol{\tau})_\theta = \frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{\theta\phi}}{\partial \phi} + \frac{1}{r}\left(\tau_{r\theta} + 2\tau_{\theta r} + \cot\theta(\tau_{\theta\theta} - \tau_{\phi\phi})\right),$$

$$(\boldsymbol{\nabla}\cdot\boldsymbol{\tau})_\phi = \frac{\partial \tau_{\phi r}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\phi\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{\phi\phi}}{\partial \phi} + \frac{1}{r}\left(\tau_{r\phi} + 2\tau_{\phi r} + \cot\theta(\tau_{\theta\phi} + \tau_{\phi\theta})\right).$$