Indian Institute of Science
ME 242: Midsemester Test

Date: 28/11/20.
Duration: 9.30 a.m.–12.00 noon
Maximum Marks: 100

1. Let $\boldsymbol{u}$ and $\boldsymbol{v}$ be unit vectors. Given that the tensor (30)

$$\boldsymbol{R} = \boldsymbol{I} + r\boldsymbol{v} \otimes \boldsymbol{u} - s(\boldsymbol{u} \otimes \boldsymbol{u} + \boldsymbol{u} \otimes \boldsymbol{v} + \boldsymbol{v} \otimes \boldsymbol{v}), \tag{1}$$

is a rotation that rotates $\boldsymbol{u}$ into $\boldsymbol{v}$,

   (a) Show that the axis of $\boldsymbol{R}$ is perpendicular to the plane containing $\boldsymbol{u}$ and $\boldsymbol{v}$. Derive any results that you require on the way.

   (b) Determine the scalars r and s as functions of $\boldsymbol{u}$ and $\boldsymbol{v}$ using the fact that $\boldsymbol{v} = \boldsymbol{R}\boldsymbol{u}$. (Hint: $\boldsymbol{u}$ and $\boldsymbol{v}$ are linearly independent vectors).

2. For the two-dimensional flow around a point vortex of constant strength Γ, where each particle moves along a circle about the origin, the equation of motion $\boldsymbol{\chi}(\boldsymbol{X}, t)$ is given by (40)

$$\begin{aligned} x &= f(\boldsymbol{X}) \cos \frac{\Gamma t}{2\pi(X^2 + Y^2)} - Y \sin \frac{\Gamma t}{2\pi(X^2 + Y^2)}, \\ y &= X \sin \frac{\Gamma t}{2\pi(X^2 + Y^2)} + g(\boldsymbol{X}) \cos \frac{\Gamma t}{2\pi(X^2 + Y^2)}. \end{aligned} \tag{2}$$

If $\boldsymbol{X} = (X, Y)$, then

   (a) Determine the scalar-valued functions $f(\boldsymbol{X})$ and $g(\boldsymbol{X})$.

   (b) Determine the velocity vector $\boldsymbol{v}(\boldsymbol{x}, t)$, the velocity gradient $\boldsymbol{L}$, and the acceleration vector $\boldsymbol{a}(\boldsymbol{x}, t)$ in the Eulerian setting. (Hint: See if the expressions for v_x and v_y have a y and x, respectively, in the numerator).

   (c) Indicate (without actually carrying out the computation) how you would find the polar components of the Eulerian velocity and acceleration, (v_r, v_θ) and (a_r, a_θ), respectively, in terms of (r, θ).

Note that you *do not* need any inputs from your fluid mechanics class (specifically, you do not need to know what a 'point vortex' means) for solving this problem. Just a basic knowledge of kinematics is sufficient.

3. Let $\phi(x, y)$ be a scalar-valued function. All the operators and tensors in this problem are two dimensional, e.g., $\boldsymbol{\nabla}^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}$, $\boldsymbol{I}$ has the components δ_{ij}, $i, j = 1, 2$, etc. All the usual rules of indicial notation are still valid. (30)

   (a) If

$$\boldsymbol{\tau} = (\boldsymbol{\nabla}^2\phi)\boldsymbol{I} + \beta\boldsymbol{\nabla}(\boldsymbol{\nabla}\phi), \tag{3}$$

   determine the constant β such that $\boldsymbol{\nabla} \cdot \boldsymbol{\tau} = \boldsymbol{0}$.

(b) Using the condition $\boldsymbol{\nabla}^2(\operatorname{tr}\boldsymbol{\tau}) = 0$, determine the governing equation for ϕ, and write this governing equation in indicial notation.

(c) Consider a function $\phi(x, y)$ such that $\boldsymbol{\nabla}^2\phi = 0$. Does this $\phi(x, y)$ satisfy the governing equation that you derived in part (b) above? Now let

$$\begin{aligned} \boldsymbol{u} &= \boldsymbol{\nabla}\phi, \\ \boldsymbol{\epsilon} &= \frac{1}{2}\left[\boldsymbol{\nabla}\boldsymbol{u} + (\boldsymbol{\nabla}\boldsymbol{u})^T\right], \\ \boldsymbol{\tau} &= \lambda(\operatorname{tr}\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\boldsymbol{\epsilon}, \end{aligned}$$

where (λ, μ) are constants. Under the constraint $\boldsymbol{\nabla}^2\phi = 0$, determine $\boldsymbol{\nabla}\cdot\boldsymbol{\tau}$ from the above three equations.