Indian Institute of Science
ME 242: Midsemester Test

Date: 3/10/21.
Duration: 9.30 a.m.–12.30 p.m.
Maximum Marks: 100

1. Let $\boldsymbol{T}$ be a given tensor. Determine if $\boldsymbol{T}(\boldsymbol{u} \otimes \boldsymbol{u}) = (\boldsymbol{u} \otimes \boldsymbol{u})\boldsymbol{T}$ holds for an arbitrary vector $\boldsymbol{u}$. If you claim that it is true, then prove it. If not, then provide a counterexample by taking $\boldsymbol{T}$ and $\boldsymbol{u}$ to be a specific matrix and vector, respectively. Next, find the principal invariants of $\boldsymbol{T}(\boldsymbol{u} \otimes \boldsymbol{u})$ and $(\boldsymbol{u} \otimes \boldsymbol{u})\boldsymbol{T}$, and determine if both sets of invariants are the same. (30)

2. A symmetric tensor $\boldsymbol{S}$ is said to be positive definite if $\boldsymbol{u} \cdot (\boldsymbol{S}\boldsymbol{u}) > 0$ for all nonzero vectors $\boldsymbol{u}$. By making a clever choice for $\boldsymbol{u}$ (Hint: eigenvalue problem), show that if $\boldsymbol{S}$ is positive definite, then all its eigenvalues are positive. Conversely, if all the eigenvalues of $\boldsymbol{S}$ are positive, show that it is positive definite (Hint: spectral resolution). (35)

   Now determine if the following symmetric tensors are positive definite, or if they contain some parameter β, determine the range of this parameter under which the given $\boldsymbol{S}$ is positive definite:

   (a) $\boldsymbol{S} = \boldsymbol{u}_0 \otimes \boldsymbol{u}_0$ where $\boldsymbol{u}_0$ is a given vector.

   (b) $\boldsymbol{S} = -\boldsymbol{W}^2$, where $\boldsymbol{W}$ is skew-symmetric.

   (c) $\boldsymbol{S} = \boldsymbol{I} + \frac{\beta \boldsymbol{W}^2}{|\boldsymbol{w}|^2}$, where β is a real number, $\boldsymbol{W}$ is skew-symmetric, and $\boldsymbol{w}$ is its axial vector.

   (d) $\boldsymbol{S} = \sum_{n=1}^{3} e^{\lambda_i} \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i^*$, where the λ_i are real numbers (not necessarily positive), and $\{\boldsymbol{e}_i^*\}$ is an orthonormal basis.

   (e) $\boldsymbol{S} = 2\boldsymbol{e} \otimes \boldsymbol{e} - \boldsymbol{I}$, where $\boldsymbol{e}$ is an arbitrary unit vector; in this case, also determine if $\boldsymbol{S}$ is orthogonal.

3. A ring of radius R rotates with a constant angular velocity $\Omega \boldsymbol{e}_3$ as shown in Fig. 1. A rigid disc of radius a whose center is fixed and at a distance d from the center of the ring is in contact on the inner side of this ring. The rigid disc is set into rotary motion due to its contact with the ring (assume that there is no slipping between the disc and the ring at this contact point). A typical point on the disc with initial coordinates (X, Y) at $t = 0$ moves to a point with coordinates (x, y) (which are obviously functions of time) with respect to the $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ basis at the center of the ring. Treat this entire problem as two-dimensional (ignore the z-coordinate) (35)

   (a) Find the equation of motion for the disc in the form $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$ with respect to the $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ basis. Express this equation of motion in terms of matrices and vectors since it will be helpful in the subsequent parts. For example, if the expression for the velocity involves a matrix of the kind $\boldsymbol{H}^{-1}\dot{\boldsymbol{H}}$, then simply state what $\boldsymbol{H}^{-1}$ and $\dot{\boldsymbol{H}}$ are, but you *need not* carry out the matrix multiplication.

(b) Determine the deformation gradient $\boldsymbol{F}$, and the Lagrangian and small strain tensors $\boldsymbol{E}$ and $\boldsymbol{\epsilon}$ at time t.

(c) Find the Lagrangian and Eulerian velocities $(\tilde{\boldsymbol{v}}, \boldsymbol{v})$ and accelerations $(\tilde{\boldsymbol{a}}, \boldsymbol{a})$. The Eulerian acceleration $\boldsymbol{a}(\boldsymbol{x}, t)$ must be found using the expression for $\boldsymbol{v}(\boldsymbol{x}, t)$.

(d) Find the velocity gradient $\boldsymbol{L}$, and the rate of deformation $\boldsymbol{D}$.

Some relevant formulae

$$\cos(A+B) = \cos A \cos B - \sin A \sin B,$$
$$\sin(A+B) = \sin A \cos B + \sin B \cos A,$$
$$\boldsymbol{W} = |\boldsymbol{w}| \, (\boldsymbol{r} \otimes \boldsymbol{q} - \boldsymbol{q} \otimes \boldsymbol{r}), \quad (\boldsymbol{w}/\left|\boldsymbol{w}\right|, \boldsymbol{q}, \boldsymbol{r} \text{ orthonormal}),$$
$$I_2 = \frac{1}{2}\left[(\operatorname{tr} \boldsymbol{T})^2 - \operatorname{tr} \boldsymbol{T}^2\right],$$
$$\det \boldsymbol{T} = \frac{1}{6}\epsilon_{ijk}\epsilon_{pqr}T_{ip}T_{jq}T_{kr}.$$

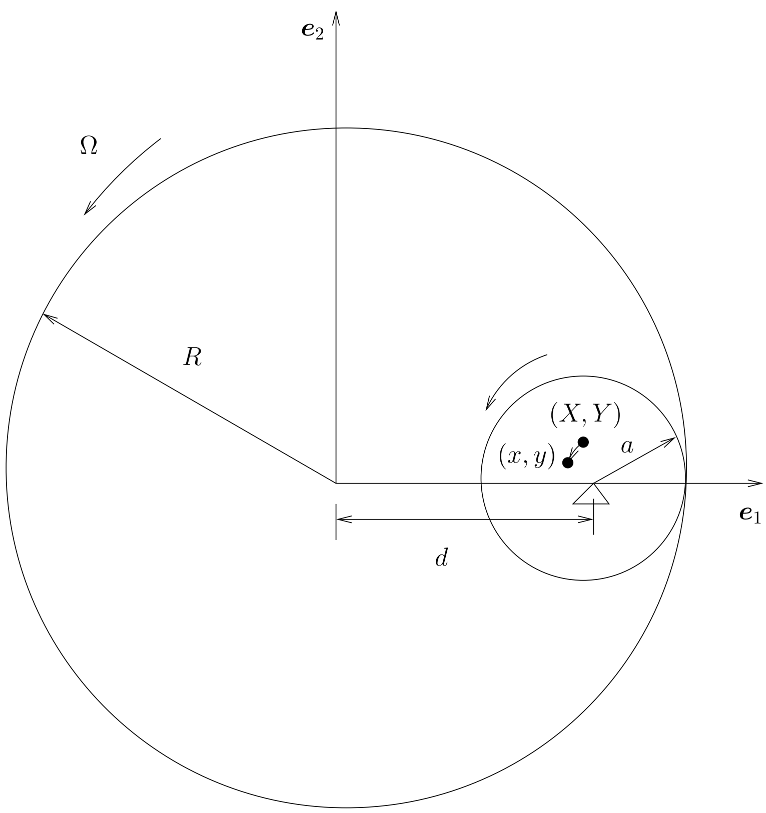

Figure 1: A rigid disc set in rotary motion due to a spinning ring.