Indian Institute of Science
ME 242: Midsemester Test

Date: 26/9/25.
Duration: 3.00 p.m.–5.30 p.m.
Maximum Marks: 100

1. Let $\boldsymbol{W} \in \text{Skw}$, and let $\boldsymbol{w}$ be its axial vector. (30)

   (a) Using indicial notation, find an expression for $\mathbf{cof}\,\boldsymbol{W}$ in terms of $\boldsymbol{w}$.

   (b) Let $\alpha > 0$ and $\beta > 0$ be two constants. Using the result of part (a), determine if

   $$\boldsymbol{T} = \alpha \mathbf{cof}\,\boldsymbol{W} + \beta \boldsymbol{W}^2,$$

   is symmetric.

   (c) Find the eigenvalues/eigenvectors of $\boldsymbol{T}$. If the eigenvectors are not unique, find one set of eigenvectors which are orthonormal.

   (d) Determine if $\boldsymbol{T}$ is positive definite.

   (e) Find $\boldsymbol{T}^{-1}$ in terms of $\boldsymbol{I}$, $\boldsymbol{w}$, α and β.

2. Let $\phi = \boldsymbol{x} \cdot \boldsymbol{v}$, where $\boldsymbol{x}$ is the position vector, and the vector field $\boldsymbol{v}$ is harmonic, i.e., $\boldsymbol{\nabla}^2 \boldsymbol{v} = \boldsymbol{0}$. It is given that (40)

   $$2\mu \boldsymbol{u} = -\boldsymbol{\nabla}\phi + 4(1-\nu)\boldsymbol{v}.$$

   Using the above information, and given that λ, μ and ν are constants, evaluate

   $$(\lambda + \mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + \mu \boldsymbol{\nabla}^2 \boldsymbol{u},$$

   as a function of $\boldsymbol{\nabla} \cdot \boldsymbol{v}$, and the constants λ, μ and ν.

3. A sphere rotates with constant angular speed ω about the Z-axis. At $t = 0$, the coordinates of a typical point in the undeformed coordinate system (X, Y, Z) are related to the spherical coordinates (R, Θ, Φ) (see Fig. 1) by the relations (30)

   $$X = R \sin\Theta \cos\Phi,$$
   $$Y = R \sin\Theta \sin\Phi,$$
   $$Z = R \cos\Theta.$$

   (a) Find the spatial coordinates (x, y, z) in terms of $(R, \Theta, \Phi, \omega, t)$ (Hint: Perturbation of some quantity by ωt).

   (b) Using the result in part (a), find $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t)$ where $\boldsymbol{X} = (X, Y, Z)$.

   (c) Find the deformation gradient, Lagrangian and Eulerian velocity and acceleration descriptions, i.e., $\boldsymbol{F}$, $\tilde{\boldsymbol{v}}$, $\boldsymbol{v}$, $\tilde{\boldsymbol{a}}$ and $\boldsymbol{a}$ (the Eulerian acceleration $\boldsymbol{a}$ should be found using the Eulerian velocity $\boldsymbol{v}$), and the rate of deformation $\boldsymbol{D}$. Is $\boldsymbol{F}$ proper orthogonal?

   (d) Find the Lagrangian strain $\boldsymbol{E}(\boldsymbol{X}, t)$ and the small strain tensor $\boldsymbol{\epsilon}(\boldsymbol{X}, t)$.

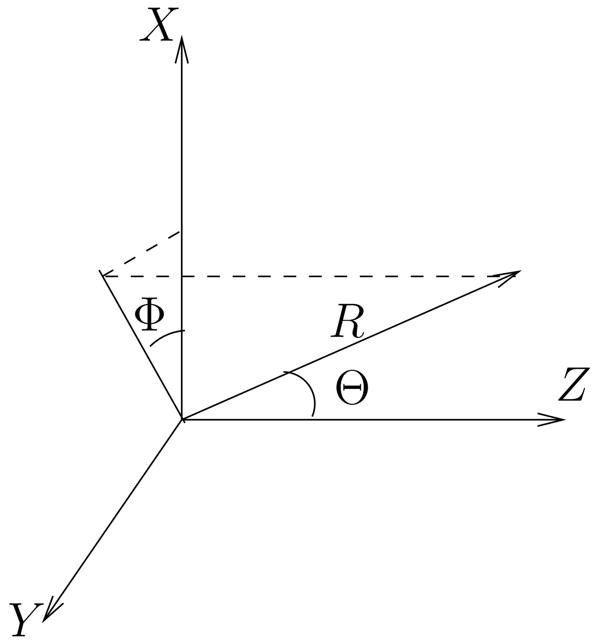

Figure 1: Spherical coordinates.

Some relevant formulae

$$
\begin{aligned}
w_i &= -\frac{1}{2}\epsilon_{ijk}W_{jk}, \\
W_{ij} &= -\epsilon_{ijk}w_k, \\
\boldsymbol{W} &= |\boldsymbol{w}|\,(\boldsymbol{r}\otimes\boldsymbol{q}-\boldsymbol{q}\otimes\boldsymbol{r}), \quad (\boldsymbol{w}/\,|\boldsymbol{w}|\,,\boldsymbol{q},\boldsymbol{r} \text{ orthonormal}), \\
(\mathbf{cof}\,\boldsymbol{T})_{ij} &= \frac{1}{2}\epsilon_{imn}\epsilon_{jpq}T_{mp}T_{nq}, \\
(\mathbf{cof}\,\boldsymbol{T})^T\boldsymbol{T} &= (\det\boldsymbol{T})\boldsymbol{I}.
\end{aligned}
$$

$$
\begin{aligned}
\cos(A+B) &= \cos A\cos B - \sin A\sin B, \\
\sin(A+B) &= \sin A\cos B + \sin B\cos A,
\end{aligned}
$$