Indian Institute of Science
ME 242: Midsemester Test

Date: 27/9/03.
Duration: 9.30 a.m.–11.00 a.m.
Maximum Marks: 100

Instructions:

1. State all your assumptions and explanations **clearly**.
2. The points for each question are indicated in the right margin.

1. This exercise seeks to answer the following question: Are there non-zero tensors with all eigenvalues zero? (30)

   (a) Taking the trace of both sides of

   $$\boldsymbol{T}^3 - I_1\boldsymbol{T}^2 + I_2\boldsymbol{T} - I_3\boldsymbol{I} = \boldsymbol{0},$$

   find an expression for $\det \boldsymbol{T}$ in terms of $\boldsymbol{T}$ and its powers.

   (b) Find the principal invariants of $\boldsymbol{a} \otimes \boldsymbol{b}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors perpendicular to each other (For finding the determinant you may use the expression derived above, or any other expression that you find convenient).

   (c) Find the eigenvalues using the characteristic equation.

   Based on the above, what answer can you provide to the question posed at the beginning of this exercise? Does this result hold for symmetric tensors? Justify your answer.

2. For the velocity field given by (25)

   $$\boldsymbol{v} = \frac{y}{1+t^3}\boldsymbol{e}_1 + t\boldsymbol{e}_2,$$

   find the acceleration of a particle occupying an initial position $\boldsymbol{X} = (X, Y)$ using the Lagrangian and Eulerian approaches, and verify that they yield the same result at $t = t_0$.

3. A circular bar of radius R and length L fixed at one end is subjected to a torque T at the other end as shown in Fig. 1. The proposed stress distribution is given by $\tau_{xz} = -G\alpha y$, $\tau_{yz} = G\alpha x$ with all other stress components zero, where G is the shear modulus, and α is a constant to be determined. (45)

   (a) Verify that the proposed stress distribution satisfies the governing differential equations, compatibility equations and traction boundary conditions at the lateral surface.

   (b) Determine the traction components (t_x, t_y, t_z) on the top surface; using these traction components, find the force components (F_x, F_y, F_z) on the top surface. (Hint: Converting from (x, y, z) to (r, θ, z) might make evaluation of the integrals involved easier in this and the following part.)

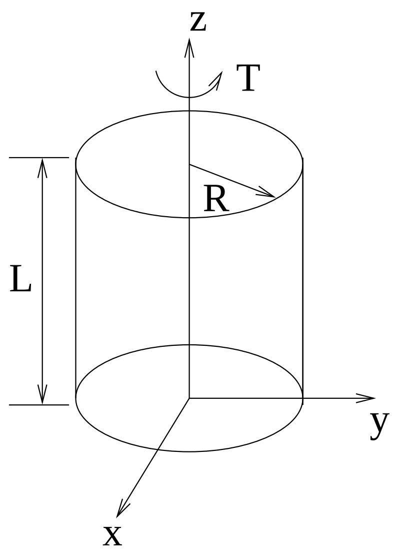

Figure 1:

(c) Using the fact that the total torque on the top surface is T, and the expressions for the traction components on the top surface, determine α.

(d) Using the relation $[\boldsymbol{v}]' = \boldsymbol{Q}[\boldsymbol{v}]$ where $(\boldsymbol{Q})_{ij} = \boldsymbol{e}_{i'} \cdot \boldsymbol{e}_j$, and the expressions for (t_x, t_y, t_z) that you have found, find the components of traction (t_r, t_θ, t_z) on the top surface.

(e) Assuming $u_z = 0$, find the displacement components (u_x, u_y) in a *systematic* way.

Some relevant formulae

$$(\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{c} \otimes \boldsymbol{d}) = (\boldsymbol{b} \cdot \boldsymbol{c})(\boldsymbol{a} \otimes \boldsymbol{d}).$$

The second and third invariants are given by

$$I_2 = \frac{1}{2}\left[(\operatorname{tr}\boldsymbol{T})^2 - \operatorname{tr}\boldsymbol{T}^2\right],$$

$$\det \boldsymbol{T} = \frac{1}{6}\epsilon_{ijk}\epsilon_{pqr}T_{ip}T_{jq}T_{kr}.$$

The equations of compatibility are

$$\frac{\partial^2 \epsilon_{xx}}{\partial y^2} + \frac{\partial^2 \epsilon_{yy}}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y},$$

$$\frac{\partial^2 \epsilon_{yy}}{\partial z^2} + \frac{\partial^2 \epsilon_{zz}}{\partial y^2} = \frac{\partial^2 \gamma_{yz}}{\partial y \partial z},$$

$$\frac{\partial^2 \epsilon_{zz}}{\partial x^2} + \frac{\partial^2 \epsilon_{xx}}{\partial z^2} = \frac{\partial^2 \gamma_{zx}}{\partial z \partial x},$$

$$2\frac{\partial^2 \epsilon_{xx}}{\partial y \partial z} = \frac{\partial}{\partial x}\left(-\frac{\partial \gamma_{yz}}{\partial x} + \frac{\partial \gamma_{xz}}{\partial y} + \frac{\partial \gamma_{xy}}{\partial z}\right),$$

$$2\frac{\partial^2 \epsilon_{yy}}{\partial z \partial x} = \frac{\partial}{\partial y}\left(-\frac{\partial \gamma_{zx}}{\partial y} + \frac{\partial \gamma_{yx}}{\partial z} + \frac{\partial \gamma_{yz}}{\partial x}\right),$$

$$2\frac{\partial^2 \epsilon_{zz}}{\partial x \partial y} = \frac{\partial}{\partial z}\left(-\frac{\partial \gamma_{xy}}{\partial z} + \frac{\partial \gamma_{zy}}{\partial x} + \frac{\partial \gamma_{zx}}{\partial y}\right).$$