Indian Institute of Science
ME 242: Midsemester Test

Date: 25/9/04.
Duration: 11.00 a.m.–12.30 p.m.
Maximum Marks: 100

Instructions:

1. Justify all your steps.
2. The points for each question are indicated in the right margin.

1. Prove the following: (25)

   (a) Let $\boldsymbol{S}$ be a symmetric tensor. Show that $(\boldsymbol{u}, \boldsymbol{S}\boldsymbol{u}) = 0$ if and only if $\boldsymbol{S} = \boldsymbol{0}$ (Hint: Make appropriate choices of $\boldsymbol{u}$).

   (b) Use the above result to prove that $\boldsymbol{Q}$ is orthogonal if and only if

$$|\boldsymbol{Q}\boldsymbol{u}| = |\boldsymbol{u}| \quad \forall \boldsymbol{u} \in V.$$

2. A relation for the determinant is (30)

$$(\boldsymbol{u} \cdot \boldsymbol{v} \times \boldsymbol{w}) \det \boldsymbol{T} = \boldsymbol{T}\boldsymbol{u} \cdot (\boldsymbol{T}\boldsymbol{v} \times \boldsymbol{T}\boldsymbol{w}). \tag{1}$$

   (a) Using Eqn. (2) (on pg. 2), *derive* an expression for $\boldsymbol{e}_j \times \boldsymbol{e}_k$ in terms of ϵ_{ijk} and $\boldsymbol{e}_i$.

   (b) Taking $\boldsymbol{u} = \boldsymbol{e}_j \times \boldsymbol{e}_k$, $\boldsymbol{v} = \boldsymbol{e}_j$ and $\boldsymbol{w} = \boldsymbol{e}_k$ in Eqn. (1), find a formula for $\det \boldsymbol{T}$ which involves only the components of $\boldsymbol{T}$, and the alternate tensor 'ϵ'.

   (c) Use this formula for $\det \boldsymbol{T}$ to find a relation between $\det \boldsymbol{T}$ and $\det \boldsymbol{T}^t$.

3. The following problem may be treated as a *two-dimensional* problem in the r-θ plane, i.e., $\tau_{rz} = \tau_{\theta z} = \tau_{zz} = 0$, and hence, for simplicity, you may write the components of $\boldsymbol{\tau}$ (which are independent of z) in the $\boldsymbol{e}_r$-$\boldsymbol{e}_\theta$ coordinate system as the 2×2 matrix $\begin{bmatrix} \tau_{rr} & \tau_{r\theta} \\ \tau_{r\theta} & \tau_{\theta\theta} \end{bmatrix}$. A hollow disc with inner radius r_1 and outer radius r_2 is subjected to a uniform pressure p at the inner boundary, as shown in Fig. 1. The solution for the components τ_{rr} and $\tau_{\theta\theta}$ is given by (45)

$$\tau_{rr} = \frac{a}{r^2} + 2b,$$
$$\tau_{\theta\theta} = -\frac{a}{r^2} + 2b,$$

where a and b are constants to be determined (in part (c) below).

   (a) Substitute the above expressions into the equilibrium equations,

$$\frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r} = 0,$$
$$r\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial (r^2 \tau_{r\theta})}{\partial r} = 0,$$

   and find an expression for $\tau_{r\theta}$.

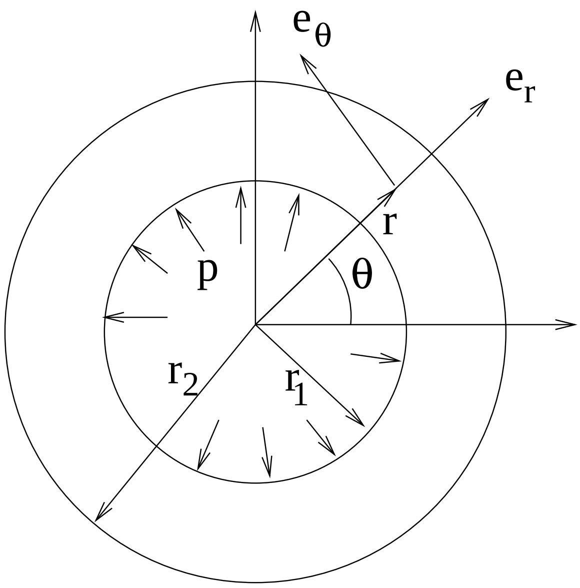

Figure 1: Disc subjected to internal pressure p.

(b) Write the traction boundary conditions (referred to the $\boldsymbol{e}_r$-$\boldsymbol{e}_\theta$ system) on the inner and outer surface of the disc, i.e., at $r = r_1$ and at $r = r_2$.

(c) Use these traction boundary conditions to determine the constants in the expressions for τ_{rr}, $\tau_{\theta\theta}$ and $\tau_{r\theta}$.

(d) Find $(\tau_{\theta\theta})_{\max}$, and the location where it occurs.

Some relevant formulae

$$\begin{aligned} \epsilon_{ijk} &= \boldsymbol{e}_i \cdot (\boldsymbol{e}_j \times \boldsymbol{e}_k), \\ \epsilon_{ijk}\epsilon_{imn} &= \delta_{jm}\delta_{kn} - \delta_{jn}\delta_{km}, \\ \boldsymbol{T}\boldsymbol{e}_j &= T_{ij}\boldsymbol{e}_i. \end{aligned} \tag{2}$$

$$(\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{c} \otimes \boldsymbol{d}) = (\boldsymbol{b} \cdot \boldsymbol{c})(\boldsymbol{a} \otimes \boldsymbol{d}).$$