Indian Institute of Science
ME 242: Midsemester Test

Date: 24/9/05.
Duration: 9.30 a.m.–11.00 p.m.
Maximum Marks: 100

Instructions:

1. Justify all your steps.
2. The points for each question are indicated in the right margin.

1. Show using indicial notation that $\operatorname{tr}(\boldsymbol{RS}) = \operatorname{tr}(\boldsymbol{SR})$ and $\det(\boldsymbol{RS}) = (\det \boldsymbol{R})(\det \boldsymbol{S})$. Use these results to show that the eigenvalues of $\boldsymbol{RS}$ and $\boldsymbol{SR}$ are the same. (25)

2. Let the velocity field be given by $\boldsymbol{v} = (\boldsymbol{x}/|\boldsymbol{x}|)$, where $\boldsymbol{x}$ is the spatial coordinate. Find the acceleration $\boldsymbol{a}$ using the Eulerian framework. (25)

3. The goal of this problem is to derive an "energy" equation over the domain V from the governing equations. (30)

   (a) If $\boldsymbol{\tau}$ is the stress tensor and $\boldsymbol{u}$ is the displacement vector, both defined over V, derive a relation between $\boldsymbol{\nabla} \cdot (\boldsymbol{\tau}^t \boldsymbol{u})$, $\boldsymbol{\tau} : \boldsymbol{\nabla u}$ and $\boldsymbol{u} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{\tau})$ (where the gradients are with respect to $\boldsymbol{x}$).

   (b) Using the symmetry of $\boldsymbol{\tau}$, find a relation between $\boldsymbol{\tau} : \boldsymbol{\nabla u}$ and $\boldsymbol{\tau} : \boldsymbol{\epsilon}(\boldsymbol{u})$, where $\boldsymbol{\epsilon}(\boldsymbol{u}) := (\boldsymbol{\nabla u} + (\boldsymbol{\nabla u})^t)/2$, and substitute it into the relation derived in (a).

   (c) Substitute the final relation obtained in (b) into the relation

   $$\int_V \boldsymbol{u} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{\tau} + \rho \boldsymbol{b}) dV = 0,$$

   and use the definition of transpose, the divergence theorem and the Cauchy relation to find a relation between the "strain energy" $\frac{1}{2} \int_V \boldsymbol{\tau} : \boldsymbol{\epsilon}(\boldsymbol{u})\, dV$, and the "work done by the forces" given by $\int_S \boldsymbol{t} \cdot \boldsymbol{u}\, dS + \int_V \rho \boldsymbol{b} \cdot \boldsymbol{u}\, dV$, where $\boldsymbol{t}$ is the traction vector.

4. A hollow cylinder with inner radius r_1 and outer radius r_2 under "plane strain" conditions (i.e., $u_z = 0$) is subjected to a uniform pressure p at the inner boundary, as shown in Fig. 1. The displacement solution is given by (20)

   $$u_r = c_1 r + \frac{c_2}{r}, \quad u_\theta = 0, \quad u_z = 0,$$

   where c_1 and c_2 are constants to be determined (in part (c) below).

   (a) Find the strains using the relations

   $$\epsilon_{rr} = \frac{\partial u_r}{\partial r}; \quad \epsilon_{zz} = \frac{\partial u_z}{\partial z}; \quad \epsilon_{\theta\theta} = \frac{u_r}{r},$$

   with the other strain components zero.

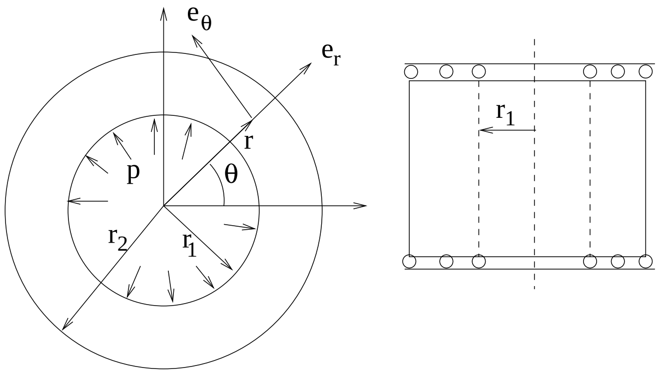

Figure 1: Cylinder subjected to internal pressure p under plane strain conditions.

(b) Using the constitutive relation $\boldsymbol{\tau} = \lambda(\mathrm{tr}\,\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\boldsymbol{\epsilon}$, find expressions for the components of $\boldsymbol{\tau}$.

(c) Write the traction boundary conditions (referred to the $\boldsymbol{e}_r$-$\boldsymbol{e}_\theta$-$\boldsymbol{e}_z$ system) on the inner and outer surface of the cylinder, i.e., at $r = r_1$ and at $r = r_2$, and on the top surface.

(d) Use these traction boundary conditions to determine the stress field.

(e) Find the *total* axial force F_z exerted on the top wall by the cylinder.

Some relevant formulae

$$\det \boldsymbol{T} = \epsilon_{ijk} T_{i1} T_{j2} T_{k3} = \epsilon_{ijk} T_{1i} T_{2j} T_{3k},$$
$$\epsilon_{pqr}(\det \boldsymbol{T}) = \epsilon_{ijk} T_{ip} T_{jq} T_{kr} = \epsilon_{ijk} T_{pi} T_{qj} T_{rk}.$$