Indian Institute of Science
ME 242: Midsemester Test

Date: 29/9/07.
Duration: 9.30 a.m.–11.00 p.m.
Maximum Marks: 100

Instructions:

1. Justify all your steps.
2. The points for each question are indicated in the right margin.

1. You have shown that $\dot{\boldsymbol{Q}}\boldsymbol{Q}^T = \boldsymbol{W}(t) \in$ Skw. If $\boldsymbol{w}(t)$ denotes the axial vector of $\boldsymbol{W}(t)$, then using the relation $\boldsymbol{x} = \boldsymbol{Q}(t)\boldsymbol{X} + \boldsymbol{c}(t)$ for a rigid motion, find an expression for the acceleration $\boldsymbol{a}(\boldsymbol{x}, t)$, and use it to find an expression for $\boldsymbol{a}(\boldsymbol{x}_1, t) - \boldsymbol{a}(\boldsymbol{x}_2, t)$ in terms of $\boldsymbol{w}$, $\dot{\boldsymbol{w}}$ and $\boldsymbol{x}_1 - \boldsymbol{x}_2$. (20)

2. A uniform radial traction p is applied to the lateral surface of a circular cylinder of radius a and length $2L$ made of an isotropic material as shown in Fig. 1a. By assuming the body forces to be zero, and the displacement field to be given by (35)

$$u_r = c_1 r + d_1 r^2,$$
$$u_z = c_2 z,$$

use the relevant differential equations and boundary conditions to find the displacement field in terms of E and ν.

3. Assume that a body is made of a linear elastic material $\tau_{ij} = \mathsf{C}_{ijkl}\epsilon_{kl}$, where $\mathbf{C}$ is symmetric, i.e., $\mathsf{C}_{ijkl} = \mathsf{C}_{klij}$. Let $(\boldsymbol{u}^{(1)}, \boldsymbol{\epsilon}^{(1)}, \boldsymbol{\tau}^{(1)}, \boldsymbol{t}^{(1)})$ and $(\boldsymbol{u}^{(2)}, \boldsymbol{\epsilon}^{(2)}, \boldsymbol{\tau}^{(2)}, \boldsymbol{t}^{(2)})$ denote the displacement, strain, stress and traction fields (which satisfy all the equations of equilibrium and boundary conditions) under two different loading conditions on the *same domain* V with surface S. Let the body forces in these two loading conditions be denoted by $\boldsymbol{b}^{(1)}$ and $\boldsymbol{b}^{(2)}$, respectively. (45)

   (a) Find a relation between $\int_S \boldsymbol{t}^{(1)} \cdot \boldsymbol{u}^{(2)}\, dS + \int_V \rho \boldsymbol{b}^{(1)} \cdot \boldsymbol{u}^{(2)}\, dV$ and $\int_V \boldsymbol{\tau}^{(1)} : \boldsymbol{\epsilon}(\boldsymbol{u}^{(2)})\, dV$.

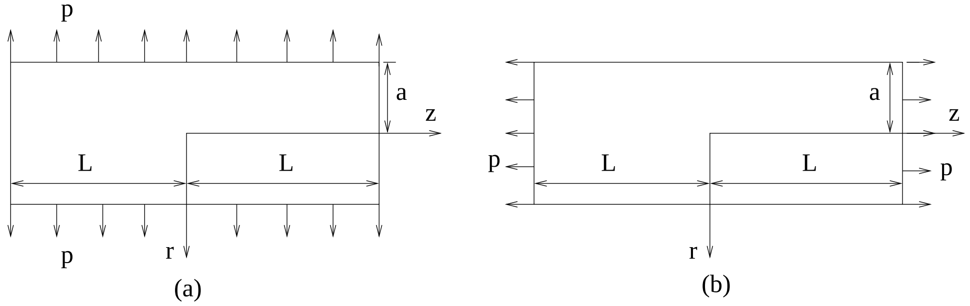

Figure 1: Problems 2 and 3

(b) Interchange the indices (1) and (2), and write the corresponding relation between $\int_S \boldsymbol{t}^{(2)} \cdot \boldsymbol{u}^{(1)}\, dS + \int_V \rho \boldsymbol{b}^{(2)} \cdot \boldsymbol{u}^{(1)}\, dV$ and $\int_V \boldsymbol{\tau}^{(2)} : \boldsymbol{\epsilon}(\boldsymbol{u}^{(1)})\, dV$.

(c) Using the symmetry of $\mathbf{C}$, find a relation between $\int_V \boldsymbol{\tau}^{(1)} : \boldsymbol{\epsilon}(\boldsymbol{u}^{(2)})\, dV$, and $\int_V \boldsymbol{\tau}^{(2)} : \boldsymbol{\epsilon}(\boldsymbol{u}^{(1)})\, dV$, and hence between $\int_S \boldsymbol{t}^{(1)} \cdot \boldsymbol{u}^{(2)}\, dS + \int_V \rho \boldsymbol{b}^{(1)} \cdot \boldsymbol{u}^{(2)}\, dV$ and $\int_S \boldsymbol{t}^{(2)} \cdot \boldsymbol{u}^{(1)}\, dS + \int_V \rho \boldsymbol{b}^{(2)} \cdot \boldsymbol{u}^{(1)}\, dV$.

(d) Using the above relation, and the solution for loading under uniform uniaxial tension p for an isotropic material under zero body forces (see Fig. 1b) given by

$$\begin{aligned} u_r &= -\frac{\nu p r}{E}, \\ u_z &= \frac{p z}{E}, \\ \tau_{zz} &= p, \end{aligned}$$

with the other stress components zero, find the displacement of the right face (i.e., $z = L$) for the loading shown in Fig. 1a.

Some relevant formulae

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\tau}^T \boldsymbol{u}) = \boldsymbol{\tau} : \boldsymbol{\epsilon}(\boldsymbol{u}) + \boldsymbol{u} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{\tau}), \quad \boldsymbol{\tau} \in \text{Sym}.$$

$$\begin{aligned} \epsilon_{rr} &= \frac{\partial u_r}{\partial r}, & \epsilon_{r\theta} &= \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right], \\ \epsilon_{\theta\theta} &= \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right), & \epsilon_{\theta z} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right), \\ \epsilon_{zz} &= \frac{\partial u_z}{\partial z}, & \epsilon_{rz} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right). \end{aligned}$$

$$\begin{aligned} (\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_r &= \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r}, \\ (\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_\theta &= \frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r}, \\ (\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_z &= \frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{z\theta}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_{zr}}{r}. \end{aligned}$$

$$\begin{bmatrix} \tau_{xx} \\ \tau_{yy} \\ \tau_{zz} \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{xz} \end{bmatrix} = h \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1-2\nu}{2} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{zz} \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{xz} \end{bmatrix},$$

where $h = E/((1+\nu)(1-2\nu))$.