Indian Institute of Science
ME 242: Midsemester Test

Date: 27/9/08.
Duration: 9.30 a.m.–11.00 p.m.
Maximum Marks: 100

Instructions:

1. Justify all your steps.
2. The points for each question are indicated in the right margin.

1. If $(\lambda_1, \boldsymbol{e}_1^*)$, $(\lambda, \boldsymbol{e}_2^*)$, $(\lambda, \boldsymbol{e}_3^*)$ are the eigenvalue/eigenvector pairs of $\boldsymbol{S}_1 \in \mathrm{Sym}$, and $(\mu_1, \boldsymbol{e}_1^*)$, $(\mu, \boldsymbol{f}_2^*)$, $(\mu, \boldsymbol{f}_3^*)$ are the eigenvalue/eigenvector pairs of $\boldsymbol{S}_2 \in \mathrm{Sym}$, determine if $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ commute, i.e., determine if $\boldsymbol{S}_1\boldsymbol{S}_2 = \boldsymbol{S}_2\boldsymbol{S}_1$. (20)

2. If $\boldsymbol{w}$ is the axial vector of $\boldsymbol{W} \in \mathrm{Skw}$, then using indicial notation, find a relation between $\mathbf{cof}\,\boldsymbol{W}$ and $\boldsymbol{w}$. Use this expression to find the second principal invariant of $\boldsymbol{W}$ in terms of $\boldsymbol{w}$. (20)

3. Let $\boldsymbol{w}(\boldsymbol{x}, t)$ be the axial vector of $\boldsymbol{W}(\boldsymbol{x}, t) \in \mathrm{Skw}$. (25)

   (a) Find an expression for $\boldsymbol{\nabla} \times \boldsymbol{W}$ in terms of $\boldsymbol{w}$.

   (b) Let $\boldsymbol{v}$ represent the velocity vector. Find a relation between $[\boldsymbol{\nabla}\boldsymbol{v} - (\boldsymbol{\nabla}\boldsymbol{v})^T]$ and $\boldsymbol{\nabla} \times \boldsymbol{v}$.

   (c) Using the above two results, find a relation of the form

   $$\boldsymbol{\nabla} \times [\boldsymbol{\nabla}\boldsymbol{v} - (\boldsymbol{\nabla}\boldsymbol{v})^T] = \alpha \boldsymbol{\nabla}(\boldsymbol{\nabla} \times \boldsymbol{v}),$$

   where you have to determine α.

4. A hollow cylinder of length L and inner and outer radii a and b, is fixed at $r = a$, and subjected to a uniform circumferential shear at $r = b$ resulting in a net moment M, as shown in Fig. 1. By assuming the body forces to be zero, and the displacement field to be given by (35)

   $$\begin{aligned} u_r &= c_1 r, \\ u_\theta &= c_2 r + \frac{c_3}{r}, \\ u_z &= 0, \end{aligned}$$

   use the relevant differential equations and boundary conditions to find the displacement, strain and stress fields in terms of the Lame constants, M and other given data. (Make sure that you show that the relevant boundary conditions are satisfied on *each* surface of the cylinder including the top and bottom ones.)

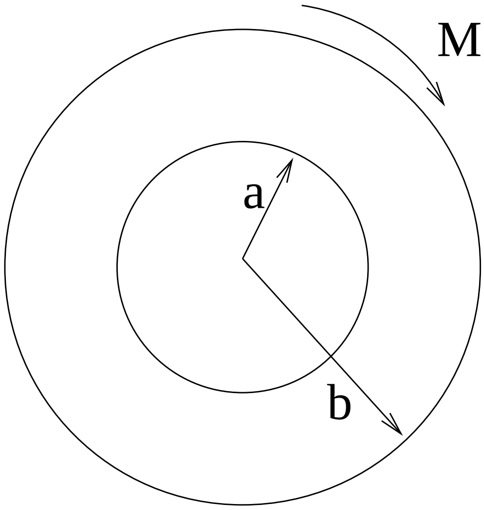

Figure 1: Hollow cylinder of length L fixed at the inner boundary $r = a$, and subjected to a moment M at the outer boundary $r = b$.

Some relevant formulae

$$
\begin{aligned}
(\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{c} \otimes \boldsymbol{d}) &= (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a} \otimes \boldsymbol{d}, \\
w_i &= -\frac{1}{2}\epsilon_{ijk} W_{jk}, \\
W_{ij} &= -\epsilon_{ijk} w_k, \\
(\mathbf{cof}\, \boldsymbol{T})_{ij} &= \frac{1}{2}\epsilon_{imn}\epsilon_{jpq} T_{mp} T_{nq}, \\
(\boldsymbol{\nabla} \times \boldsymbol{T})_{ij} &= \epsilon_{irs}\frac{\partial T_{js}}{\partial x_r}.
\end{aligned}
$$

$$
\begin{aligned}
\epsilon_{rr} &= \frac{\partial u_r}{\partial r}, & \epsilon_{r\theta} &= \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right], \\
\epsilon_{\theta\theta} &= \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right), & \epsilon_{\theta z} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right), \\
\epsilon_{zz} &= \frac{\partial u_z}{\partial z}, & \epsilon_{rz} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right).
\end{aligned}
$$

$$
\boldsymbol{\tau} = \lambda(\operatorname{tr}\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\boldsymbol{\epsilon}.
$$

$$
\begin{aligned}
(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_r &= \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r}, \\
(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_\theta &= \frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r}, \\
(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_z &= \frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{z\theta}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_{zr}}{r}.
\end{aligned}
$$