Indian Institute of Science
ME 242: Midsemester Test

Date: 23/9/09.
Duration: 4.00 p.m.–5.30 p.m.
Maximum Marks: 100

Instructions:

1. Justify all your steps.

2. The points for each question are indicated in the right margin.

1. Answer the following: (25)

   (a) If $\beta(\boldsymbol{a} \times \boldsymbol{b})$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are vectors, and β is a scalar, is the axial vector of the skew tensor $\boldsymbol{b} \otimes \boldsymbol{a} - \boldsymbol{a} \otimes \boldsymbol{b}$, find β.

   (b) Using the relation $\boldsymbol{W}\boldsymbol{u} = \boldsymbol{w} \times \boldsymbol{u}$, find an expression for $\boldsymbol{W}_1\boldsymbol{W}_2$ in terms of the axial vectors $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$ of the skew-symmetric tensors $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$. *Derive* any expressions that you need on the way.

   (c) Using the above results, find the axial vector of $\boldsymbol{W}_1\boldsymbol{W}_2 - \boldsymbol{W}_2\boldsymbol{W}_1$.

2. Let $\phi(\boldsymbol{x}, t)$ and $\boldsymbol{f}(\boldsymbol{x}, t)$ be scalar and vector-valued field variables. Given that the displacement field $\boldsymbol{u} = \boldsymbol{\nabla}\phi + \boldsymbol{\nabla} \times \boldsymbol{f}$ satisfies the equation of motion under zero body force given by (40)

$$(\lambda + \mu)\boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) + \mu\boldsymbol{\nabla}^2\boldsymbol{u} = \rho\frac{\partial^2\boldsymbol{u}}{\partial t^2},$$

   where

$$\boldsymbol{\nabla}^2\phi = \frac{\rho}{\lambda + 2\mu}\frac{\partial^2\phi}{\partial t^2}, \quad \boldsymbol{\nabla}^2\boldsymbol{f} = \frac{1}{c_2^2}\frac{\partial^2\boldsymbol{f}}{\partial t^2},$$

   find c_2^2 in terms of the Lame constants and ρ. (Hint: Since $\boldsymbol{x}$ and t are independent variables, you can interchange the order of differentiation between x_i and t.)

3. A hollow cylinder of length L and inner and outer radii a and b, is fixed at $r = a$, and subjected to a uniform shear stress at $r = b$ resulting in a net shear force T, as shown in Fig. 1. By assuming the body forces to be zero, and the displacement field to be given by (35)

$$\begin{aligned} u_r &= k_1 + k_2 r, \\ u_\theta &= 0, \\ u_z &= c_1 + c_2 \log r + c_3 z, \end{aligned}$$

   use the relevant differential equations and boundary conditions to find the displacement, strain and stress fields in terms of the Lame constants, T and other given data. (Make sure that you show that the relevant boundary conditions are satisfied on *each* surface of the cylinder including the top and bottom ones.)

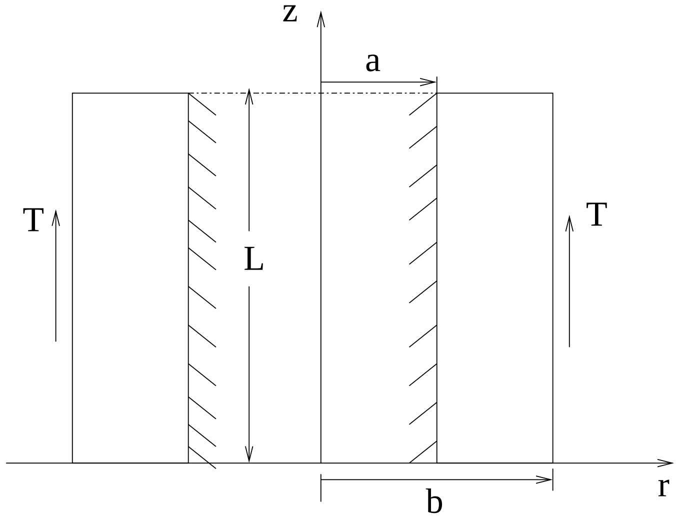

Figure 1: Hollow cylinder of length L fixed at the inner boundary $r = a$, and subjected to a shear force T at the outer boundary $r = b$.

Some relevant formulae

$$
\begin{aligned}
(\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{c} \otimes \boldsymbol{d}) &= (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a} \otimes \boldsymbol{d}, \\
w_i &= -\frac{1}{2}\epsilon_{ijk} W_{jk}, \\
W_{ij} &= -\epsilon_{ijk} w_k,
\end{aligned}
$$

$$
\begin{aligned}
\epsilon_{rr} &= \frac{\partial u_r}{\partial r}, & \epsilon_{r\theta} &= \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right], \\
\epsilon_{\theta\theta} &= \frac{1}{r}\left(\frac{\partial u_\theta}{\partial \theta} + u_r\right), & \epsilon_{\theta z} &= \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right), \\
\epsilon_{zz} &= \frac{\partial u_z}{\partial z}, & \epsilon_{rz} &= \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right).
\end{aligned}
$$

$$
\boldsymbol{\tau} = \lambda(\operatorname{tr}\boldsymbol{\epsilon})\boldsymbol{I} + 2\mu\boldsymbol{\epsilon}.
$$

$$
\begin{aligned}
(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_r &= \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r}, \\
(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_\theta &= \frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r}, \\
(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_z &= \frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{z\theta}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_{zr}}{r}.
\end{aligned}
$$