Indian Institute of Science, Bangalore

ME 243: Endsemester Exam

Date: 12/12/98.
Duration: 9.00 p.m.–12.00 a.m.
Maximum Marks: 200

1. Let $\boldsymbol{x}(\boldsymbol{x}, t)$ be a motion and $\boldsymbol{v}$ the corresponding velocity field. Assuming that $\boldsymbol{v}$ is just once-differentiable, prove that the following characterizations of rigid motion are equivalent: (40)
(i) At each time t, $\boldsymbol{v}$ admits the representation
$$\boldsymbol{v}(\boldsymbol{x}_1, t) = \boldsymbol{v}(\boldsymbol{x}_2, t) + \boldsymbol{W}(t)(\boldsymbol{x}_1 - \boldsymbol{x}_2),$$
where $\boldsymbol{W} \in \text{Skw}$.
(ii) The rate-of-deformation tensor, $\boldsymbol{D}(\boldsymbol{x}, t) = \boldsymbol{0}$, for all $(\boldsymbol{x}, t)$.
(Hint: Prove that $(\boldsymbol{x}^1 - \boldsymbol{x}^2) \cdot [\boldsymbol{v}(\boldsymbol{x}^1, t) - \boldsymbol{v}(\boldsymbol{x}^2, t)] = 0 \quad \forall \boldsymbol{x}^1, \boldsymbol{x}^2 \in V$.)

2. Derive the relation (20)
$$\frac{D\boldsymbol{F}}{Dt} = \boldsymbol{L}\boldsymbol{F}.$$
Next, using the relation $\boldsymbol{C} = \boldsymbol{F}^t\boldsymbol{F}$, relate $d\boldsymbol{C}/dt$ and $\boldsymbol{D}$. Using $dJ/dt = J\operatorname{tr}\boldsymbol{D}$, derive the relation
$$\frac{DJ}{Dt} = \frac{1}{2}J\boldsymbol{C}^{-1} : \dot{\boldsymbol{C}}.$$

3. Our goal is to derive the referential (or material) form of the energy equation given by (60)
$$\rho\frac{De}{Dt} = \boldsymbol{\tau} : \boldsymbol{L} - \boldsymbol{\nabla} \cdot \boldsymbol{q} + \rho Q_h.$$
Towards this end, we proceed as follows:

   - For arbitrary $\boldsymbol{T} \in \text{Lin}$ and $\boldsymbol{v} \in V$, derive the relation
   $$\boldsymbol{\nabla} \cdot (\boldsymbol{T}^t\boldsymbol{v}) = \boldsymbol{T} : \boldsymbol{\nabla}\boldsymbol{v} + \boldsymbol{v} \cdot (\boldsymbol{\nabla} \cdot \boldsymbol{T}).$$
   - Define the Piola transform for a vector field as follows:
   Given a vector field $\boldsymbol{v} : V \to \Re^3$ over the deformed configuration V, its Piola transform is the vector field $\boldsymbol{u} : V_0 \to \Re^3$ defined over the reference configuration by the relation
   $$\boldsymbol{u}(\boldsymbol{X}) = J\boldsymbol{F}^{-1}(\boldsymbol{X})\boldsymbol{v}(\boldsymbol{x}).$$

Show that the divergences of the two vector fields are related by:

$$\boldsymbol{\nabla}_X \cdot \boldsymbol{u}(\boldsymbol{X}) = J \boldsymbol{\nabla}_x \cdot \boldsymbol{v}(\boldsymbol{x}).$$

Note that this relation is similar to the relation between the divergences of tensor fields related through a Piola transform. (You may use $\boldsymbol{\nabla}_X \cdot \mathbf{cof}\, \boldsymbol{F} = \mathbf{0}$.)

- Use the above results and the expression for the first Piola-Kirchhoff stress $\boldsymbol{T} = \boldsymbol{\tau} \mathbf{cof}\, \boldsymbol{F}$, to get the material form of the energy equation in terms of the variables ρ_0, $\boldsymbol{T}$, $\dot{\boldsymbol{F}}$, $\boldsymbol{q}^0$, and $Q_h(\boldsymbol{X})$.

4. A homogeneous material has a stored energy function of the form (40)

$$\hat{W}(\boldsymbol{F}) = \frac{a_1}{2} \boldsymbol{F} : \boldsymbol{F} + \frac{a_2}{4} \left[(\boldsymbol{F} : \boldsymbol{F})^2 - (\boldsymbol{F}\boldsymbol{F}^t) : (\boldsymbol{F}\boldsymbol{F}^t) \right] + a_3 (\det \boldsymbol{F})^2 - a_4 \ln \det \boldsymbol{F},$$

where a_1, a_2, a_3 and a_4 are positive constants.

- Is $\hat{W}(\boldsymbol{F})$ frame-indifferent and/or isotropic? Justify.
- Find the associated response function for the first Piola-Kirchhoff stress $\hat{\boldsymbol{T}}(\boldsymbol{F})$.
- Find the condition that should be satisfied by the constants a_1-a_4 so that the reference configuration is a natural state.

You may use the result

$$\det(\boldsymbol{R} + \boldsymbol{S}) = \det \boldsymbol{R} + \mathbf{cof}\, \boldsymbol{R} : \boldsymbol{S} + \boldsymbol{R} : \mathbf{cof}\, \boldsymbol{S} + \det \boldsymbol{S}.$$

5. C is *strongly elliptic* (or satisfies the *strong Legendre-Hadamard* condition) if (40)

$$\boldsymbol{A} : \mathsf{C}[\boldsymbol{A}] > 0,$$

whenever $\boldsymbol{A}$ has the form $\boldsymbol{A} = \boldsymbol{a} \otimes \boldsymbol{c}$, $\boldsymbol{a} \neq \mathbf{0}$, $\boldsymbol{c} \neq \mathbf{0}$.

(a) Show that if C is positive definite, then C is strongly elliptic.

(b) Show that for an isotropic material C is strongly elliptic if and only if

$$\mu > 0, \quad \lambda + 2\mu > 0.$$