Indian Institute of Science, Bangalore

ME 243: Endsemester Exam

Date: 8/12/08.
Duration: 9.30 a.m.–12.30 p.m.
Maximum Marks: 100

1. Let $\mathrm{Skw}[\boldsymbol{w}]$ denote the skew-symmetric tensor of which $\boldsymbol{w}$ is the axial vector, i.e. $\mathrm{Skw}[\boldsymbol{w}]\boldsymbol{u} = \boldsymbol{w} \times \boldsymbol{u}$ for all $\boldsymbol{u}$. Using indicial notation or otherwise, find a relation between $(\mathbf{cof}\,\boldsymbol{T}\,\boldsymbol{u}) \times \boldsymbol{v}$ and $\boldsymbol{T}\left(\boldsymbol{u} \times (\boldsymbol{T}^T\boldsymbol{v})\right)$, where $\boldsymbol{u}$ and $\boldsymbol{v}$ are arbitrary vectors. Use this relation to find a relation between $\mathrm{Skw}[\mathbf{cof}\,\boldsymbol{T}\,\boldsymbol{u}]$ and $\boldsymbol{T}\,\mathrm{Skw}[\boldsymbol{u}]\,\boldsymbol{T}^T$. (20)

   Let $\boldsymbol{T} \equiv \boldsymbol{Q} \in \mathrm{Orth}^+$. Using the above results, find a relation between $e^{\mathrm{Skw}[\boldsymbol{Q}\boldsymbol{u}]}$ and $e^{\mathrm{Skw}[\boldsymbol{u}]}$.

2. We want to derive an explicit formula for $\boldsymbol{R}(t) \equiv e^{\boldsymbol{W}t}$, $\boldsymbol{W} \in \mathrm{Skw}$, when the underlying space dimension is 3. To derive this formula, let $\boldsymbol{R}(t) = h_1(t)\boldsymbol{I} + h_2(t)\boldsymbol{W} + h_3(t)\boldsymbol{W}^2$, where $h_i(t)$, $i = 1, 2, 3$, are functions to be determined. Using the equation $\dot{\boldsymbol{R}} = \boldsymbol{W}\boldsymbol{R}$ and appropriate initial conditions on $\boldsymbol{R}$ and $\dot{\boldsymbol{R}}$, determine the differential equations and initial conditions for the functions $h_i(t)$. Solve these equations to find the desired explicit formula. (30)

3. A circular shaft of initial radius R_0 and length L is subjected to a torque T and net axial force N by tractions applied to the top and bottom surfaces. The lateral surfaces are traction free. The Z-axis lies along the axis of the cylinder, with the origin at the center of the bottom surface. The constitutive relation is given by (20)

$$\boldsymbol{S} = \kappa(J-1)J\boldsymbol{C}^{-1} + \mu(\det\boldsymbol{C})^{-1/3}\left[\boldsymbol{I} - \frac{\mathrm{tr}\,\boldsymbol{C}}{3}\boldsymbol{C}^{-1}\right].$$

   The deformation is assumed to be of the form

$$\begin{aligned} r &= g(R), \\ \theta &= \Theta + dZ, \\ z &= fZ. \end{aligned}$$

   Using the scale factors $h_i = (1, r, 1)$ and $h_J^0 = (1, R, 1)$, and the relation

$$F_{iJ} = \frac{h_i}{h_J^0}\frac{\partial\hat{\chi}_i}{\partial\eta_J}, \quad \text{no sum on } i,\ J,$$

   where η_J denote (R, Θ, Z), set up the equation (*but do not solve*) and boundary conditions for determining $g(R)$. Also set up the equations for determining the constants d and f.

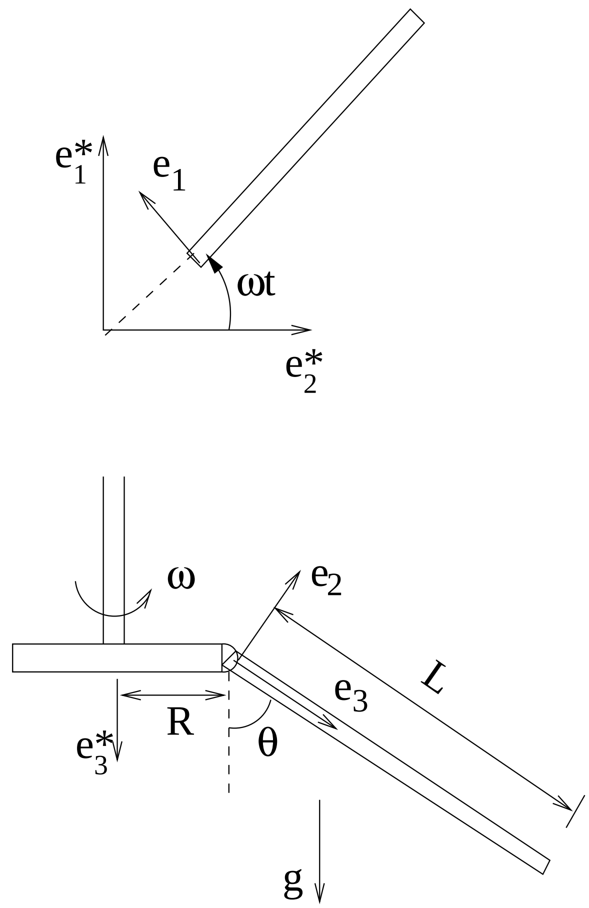

Figure 1: Rotating bar.

4. A rigid bar of length L and mass per unit length m_0 (i.e., $\rho\, dV = m_0\, dx$) is hinged to a spinning disk of radius R as shown in Fig. 1. The local axes $\{\boldsymbol{e}_i\}$ can be expressed in terms of the stationary axes $\{\boldsymbol{e}_i^*\}$ as (30)

$$\begin{aligned}
\boldsymbol{e}_1 &= \cos\omega t\boldsymbol{e}_1^* - \sin\omega t\boldsymbol{e}_2^*,\\
\boldsymbol{e}_2 &= \cos\theta(\sin\omega t\boldsymbol{e}_1^* + \cos\omega t\boldsymbol{e}_2^*) - \sin\theta\boldsymbol{e}_3^*,\\
\boldsymbol{e}_3 &= \sin\theta(\sin\omega t\boldsymbol{e}_1^* + \cos\omega t\boldsymbol{e}_2^*) + \cos\theta\boldsymbol{e}_3^*.
\end{aligned}$$

By applying the appropriate balance laws in integral form to the material volume comprised of the bar, find the angle θ made by the bar under steady state conditions, and the reaction exerted by the bar at the hinge. Assuming that the bar is vertical at $t = 0$, is the total energy (kinetic plus potential) conserved between the initial and steady-state conditions?

Some relevant formulae

$$
\begin{aligned}
(\mathbf{cof}\,\boldsymbol{T})_{ij} &= \frac{1}{2}\epsilon_{imn}\epsilon_{jpq}T_{mp}T_{nq},\\
\boldsymbol{\tau} &= \frac{1}{J}\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^T,\\
I_2(\boldsymbol{T}) &= \frac{1}{2}\left[(\mathrm{tr}\,\boldsymbol{T})^2 - \mathrm{tr}\,(\boldsymbol{T}^2)\right],\\
w_i &= -\frac{1}{2}\epsilon_{ijk}W_{jk},\\
W_{ij} &= -\epsilon_{ijk}w_k.
\end{aligned}
$$

If $\boldsymbol{\tau}$ is symmetric tensor-valued field, then the components of $\boldsymbol{\nabla}_x \cdot \boldsymbol{\tau}$ with respect to a cylindrical coordinate system are

$$
\begin{aligned}
(\boldsymbol{\nabla}\cdot\boldsymbol{\tau})_r &= \frac{\partial\tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\partial\tau_{rz}}{\partial z} + \frac{\tau_{rr}-\tau_{\theta\theta}}{r},\\
(\boldsymbol{\nabla}\cdot\boldsymbol{\tau})_\theta &= \frac{\partial\tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta} + \frac{\partial\tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r},\\
(\boldsymbol{\nabla}\cdot\boldsymbol{\tau})_z &= \frac{\partial\tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{z\theta}}{\partial\theta} + \frac{\partial\tau_{zz}}{\partial z} + \frac{\tau_{zr}}{r}.
\end{aligned}
$$