Indian Institute of Science, Bangalore

ME 243: Endsemester Exam

Date: 10/12/09.
Duration: 9.30 a.m.–12.30 p.m.
Maximum Marks: 100

1. Show that a scalar-valued function $\phi : \mathrm{Skw} \to \Re$ is isotropic if and only if there exists a function $\tilde{\phi} : \mathrm{tr}\,(\boldsymbol{W}^2) \to \Re$ such that (30)

$$\phi(\boldsymbol{W}) = \tilde{\phi}(\mathrm{tr}\,\boldsymbol{W}^2) \quad \forall \boldsymbol{W} \in \mathrm{Skw}.$$

   (Hint: Show that $\mathrm{tr}\,(\boldsymbol{W}_1^2) = \mathrm{tr}\,(\boldsymbol{W}_2^2)$ implies $\phi(\boldsymbol{W}_1) = \phi(\boldsymbol{W}_2)$.)

2. Using the velocity and pressure fields $\tilde{\boldsymbol{v}}(\boldsymbol{X}, t)$ and $\tilde{p}(\boldsymbol{X}, t)$ and the deformation gradient $\boldsymbol{F}$, develop a Lagrangian formulation for an incompressible Newtonian fluid. In particular, Lagrangian versions of the incompressibility condition $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$, momentum equation $\boldsymbol{\nabla} \cdot \boldsymbol{\tau} + \rho \boldsymbol{b} = \rho \boldsymbol{a}$, and the constitutive relation $\boldsymbol{\tau} = -p\boldsymbol{I} + 2\mu \boldsymbol{D}$ should be developed. Use this formulation to solve for the steady-state profile of a fluid in a container undergoing constant acceleration $a\boldsymbol{e}_x$ as shown in Fig. 1 (h denotes the height of the fluid in the undeformed configuration) by assuming the deformation to be of the form (40)

$$\begin{aligned} x &= X, \\ y &= Y + \gamma \left(\frac{L}{2} - X \right), \\ z &= Z, \end{aligned}$$

   and determining γ.

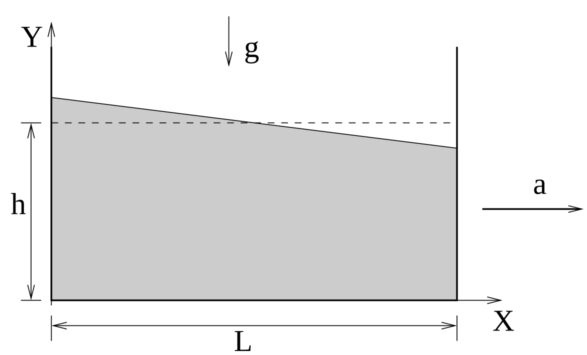

Figure 1: Fluid in a container travelling with uniform acceleration.

3. This problem attempts to find an exact solution to the bending of a prismatic beam of rectangular cross-section with width b and height h into a region bounded by two concentric arcs as shown in Fig. 2. The material is assumed to be a St Venant-Kirchhoff material with $\nu = 0$. The neutral plane (i.e., (30)

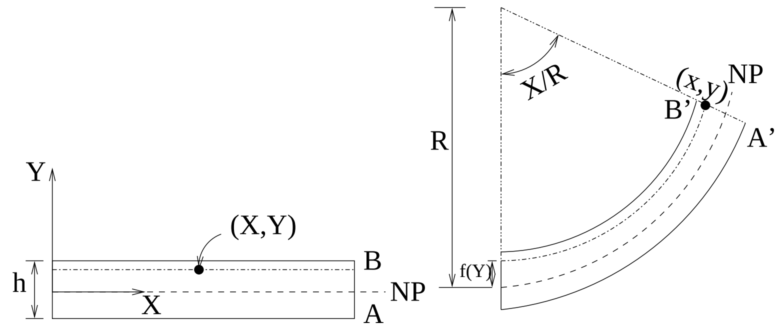

Figure 2: Bending of a prismatic beam made of a St Venant-Kirchhoff material with $\nu = 0$ into a circular arc; the neutral plane is assumed to be midway along the thickness in the undeformed configuration, i.e., at a distance $h/2$ from the top surface. The point (X, Y) deforms to (x, y) as shown.

along which $E_{XX} = 0$) is assumed to deform into an arc of radius R, and each plane Y = constant in the undeformed configuration is assumed to deform into a plane of constant radius $(R - f(Y))$, where $f(Y)$ is a function to be determined. The top and bottom surfaces are assumed to be traction free. Note that the planes X = constant are assumed to remain plane after deformation. Under these assumptions, the deformation is given by

$$r = R - f(Y), \quad \theta = \frac{X}{R}, \quad z = Z.$$

(a) Using the scale factors $h_i = (1, r, 1)$ and $h_J^0 = (1, 1, 1)$, and the relation

$$F_{iJ} = \frac{h_i}{h_J^0} \frac{\partial \hat{\chi}_i}{\partial \eta_J}, \quad \text{no sum on } i,\ J,$$

find the deformation gradient and Lagrangian strain $\boldsymbol{E}$, the second Piola-Kirchhoff stress $\boldsymbol{S} = E\boldsymbol{E}$, and hence the Cauchy stress $\boldsymbol{\tau}$.

(b) Use the equations of equilibrium in the *deformed* configuration to find the governing differential equation for $f(Y)$. *Do not* attempt to solve this equation.

(c) Find the appropriate boundary conditions for $f(Y)$.

(d) If the total moment applied at the right end about the neutral axis is M, write an equation that relates M to $f(Y)$. Again, do not evaluate any integral that arises, but your integral should be in terms of known quantities such as h, b, $f(Y)$ etc.

Some relevant formulae

$$I_2(\boldsymbol{T}) = \frac{1}{2}[(\operatorname{tr}\boldsymbol{T})^2 - \operatorname{tr}\boldsymbol{T}^2],$$
$$\boldsymbol{\tau} = \frac{1}{J}\boldsymbol{F}\boldsymbol{S}\boldsymbol{F}^T,$$
$$\boldsymbol{\nabla}_X \cdot \boldsymbol{T} = J\boldsymbol{\nabla}_x \cdot \boldsymbol{\tau},$$
$$w_i = -\frac{1}{2}\epsilon_{ijk}W_{jk},$$
$$W_{ij} = -\epsilon_{ijk}w_k,$$
$$\boldsymbol{W} = |\boldsymbol{w}|\,(\boldsymbol{r}\otimes\boldsymbol{q} - \boldsymbol{q}\otimes\boldsymbol{r}),$$
$$\boldsymbol{b} = \boldsymbol{Q}^T\,[\boldsymbol{b}^* - \ddot{\mathbf{c}}] - \dot{\boldsymbol{\Omega}}\times\boldsymbol{x} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{x}) - 2\boldsymbol{\Omega}\times\boldsymbol{v}.$$
$$\boldsymbol{b}_0(\boldsymbol{X}, t) = \boldsymbol{b}(\boldsymbol{\chi}(\boldsymbol{X}, t), t),$$
$$\rho_0 = \tilde{\rho}J.$$

If $\boldsymbol{\tau}$ is symmetric tensor-valued field, then the components of $\boldsymbol{\nabla}_x \cdot \boldsymbol{\tau}$ with respect to a cylindrical coordinate system are

$$\begin{aligned}
(\boldsymbol{\nabla}\cdot\boldsymbol{\tau})_r &= \frac{\partial\tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{r\theta}}{\partial\theta} + \frac{\partial\tau_{rz}}{\partial z} + \frac{\tau_{rr}-\tau_{\theta\theta}}{r},\\
(\boldsymbol{\nabla}\cdot\boldsymbol{\tau})_\theta &= \frac{\partial\tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{\theta\theta}}{\partial\theta} + \frac{\partial\tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r},\\
(\boldsymbol{\nabla}\cdot\boldsymbol{\tau})_z &= \frac{\partial\tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial\tau_{z\theta}}{\partial\theta} + \frac{\partial\tau_{zz}}{\partial z} + \frac{\tau_{zr}}{r}.
\end{aligned}$$