Indian Institute of Science, Bangalore

ME 243: Endsemester Exam

Date: 12/12/11.
Duration: 9.30 a.m.–12.30 p.m.
Maximum Marks: 100

Instructions:

You may directly use the formulae at the back.

1. Let the underlying vector space dimension be 3. (20)

   (a) By the spectral resolution, we have seen that any $\boldsymbol{S} \in \text{Sym}$ can be written as $\boldsymbol{S} = \sum_{i=1}^{3} \lambda_i \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i^*$? Does this imply that Sym is a three-dimensional space with $\{\boldsymbol{e}_1^* \otimes \boldsymbol{e}_1^*, \boldsymbol{e}_2^* \otimes \boldsymbol{e}_2^*, \boldsymbol{e}_3^* \otimes \boldsymbol{e}_3^*\}$ as the basis? If it is not, then specify additional elements in terms of the $\{\boldsymbol{e}_i^*\}$ so as to form a basis for Sym. Since *every* $\boldsymbol{S} \in \text{Sym}$ can be written as $\sum_{i=1}^{3} \lambda_i \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i^*$, *justify* (in case you think additional elements are required) why additional elements are required.

   (b) When $\boldsymbol{W} \in \text{Skw}$, it is obvious that $\alpha\boldsymbol{I} + \beta\boldsymbol{W}^2 \in \text{Sym}$. Conversely, given $\boldsymbol{S} \in \text{Sym}$, does there always exist a $\boldsymbol{W} \in \text{Skw}$ and $\alpha, \beta \in \Re$ such that $\boldsymbol{S} = \alpha\boldsymbol{I} + \beta\boldsymbol{W}^2$? If not, then what special class of symmetric tensors does $\alpha\boldsymbol{I} + \beta\boldsymbol{W}^2$ span? *Prove* all your statements.

2. For $\boldsymbol{T} \in \text{Lin}$, is it true that (10)

$$(e^{\boldsymbol{T}})^n = e^{n\boldsymbol{T}},$$

   for all positive and negative integers n? Justify.

3. If $\boldsymbol{W}^2\boldsymbol{T} = \boldsymbol{T}\boldsymbol{W}^2$ for all $\boldsymbol{W} \in \text{Skw}$, then deduce and prove the most general form of $\boldsymbol{T}$? Using this result or otherwise, find the most general form of $\boldsymbol{T}$ if $\boldsymbol{S}\boldsymbol{T} = \boldsymbol{T}\boldsymbol{S}$ for all $\boldsymbol{S} \in \text{Sym}$. (25)

4. A Newtonian, incompressible, viscous fluid with constitutive relation $\boldsymbol{\tau} = -p\boldsymbol{I} + 2\mu\boldsymbol{D}$ is contained in a cylindrical container of radius a as shown in Fig. 1. At time $t = 0$, the container starts rotating with an angular speed $\omega(t)\boldsymbol{e}_z$ with respect to a stationary observer as shown in Fig. 1, and ultimately reaches a steady-state value $\omega_0\boldsymbol{e}_z$. The fluid also ultimately reaches a steady-state velocity. Assume $v_z = 0$ and $\boldsymbol{b}^* = \boldsymbol{c} = \boldsymbol{0}$, throughout the motion. (20)

   (a) During the transient phase, find the body force *with respect to a rotating observer* fixed to the container using a cylindrical coordinate system. Assuming $\boldsymbol{v} = (0, v_\theta(r,t), 0)$, and $p = p(r,t)$, write the governing equations for v_θ and p (do not attempt to solve these equations). State the appropriate boundary conditions for this problem.

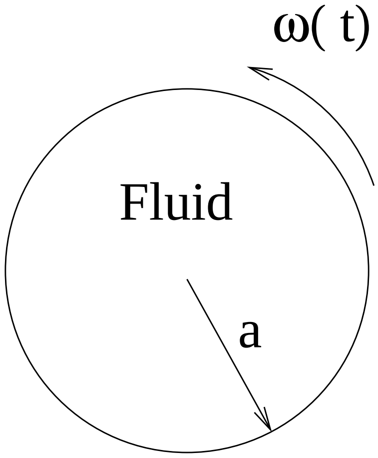

Figure 1: Problem 4.

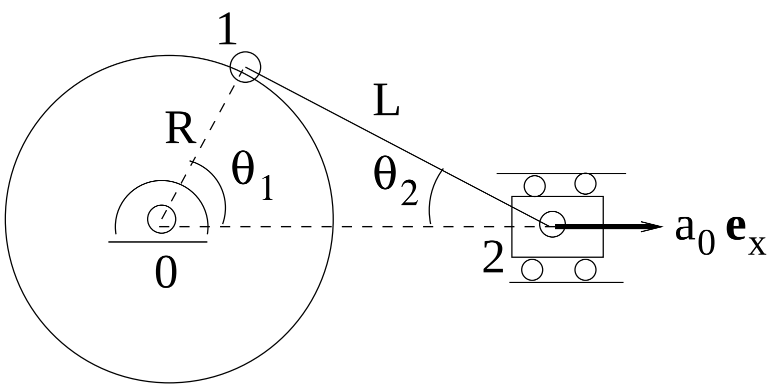

Figure 2: Slider-crank mechanism.

(b) Solve for the steady-state velocity and stress fields in the fluid with respect to the rotating observer (you may guess the solution to any complicated ODEs that you encounter).

5. Using the characterization for rigid motion given by $\boldsymbol{\chi}(\boldsymbol{X},t) = \boldsymbol{Q}(t)\boldsymbol{X} + \boldsymbol{c}(t)$, find the expressions for the velocity and acceleration when the motion is rigid. Use these results to solve the following problem *using appropriate balance laws* (do not use the Euler equations). Consider the slider-crank mechanism shown in Fig. 2, which is initially at rest. At $t = 0$, the bottom end of the connecting rod is given an acceleration $a_0\boldsymbol{e}_x$. Assuming the connecting rod to be rigid and massless, and the disc to be rigid with density ρ, find the force exerted by the connecting rod on the disc at the pin joint at the instant the acceleration $a_0\boldsymbol{e}_x$ is applied. Assume the disc to have unit width (you can treat the problem as two-dimensional). (25)

Some relevant formulae

$$
\begin{aligned}
w_i &= -\frac{1}{2}\epsilon_{ijk}W_{jk}, \\
W_{ij} &= -\epsilon_{ijk}w_k, \\
\boldsymbol{W} &= |\boldsymbol{w}|\,(\boldsymbol{r}\otimes\boldsymbol{q}-\boldsymbol{q}\otimes\boldsymbol{r}), \\
\boldsymbol{b} &= \boldsymbol{Q}^T\left[\boldsymbol{b}^*-\ddot{\boldsymbol{c}}\right]-\dot{\boldsymbol{\Omega}}\times\boldsymbol{x}-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{x})-2\boldsymbol{\Omega}\times\boldsymbol{v}. \\
\boldsymbol{\Omega} &= \begin{bmatrix}\dot{\boldsymbol{e}}_2\cdot\boldsymbol{e}_3\\ \dot{\boldsymbol{e}}_3\cdot\boldsymbol{e}_1\\ \dot{\boldsymbol{e}}_1\cdot\boldsymbol{e}_2\end{bmatrix}.
\end{aligned}
$$

<u>Continuity equation:</u>

$$
\boldsymbol{\nabla}\cdot\boldsymbol{v}=\frac{1}{r}\frac{\partial(rv_r)}{\partial r}+\frac{1}{r}\frac{\partial v_\theta}{\partial\theta}+\frac{\partial v_z}{\partial z}=0.
$$

<u>The momentum equations in r, θ and z directions:</u>

$$
\begin{aligned}
\frac{\partial v_r}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})v_r-\frac{v_\theta^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r}+\nu\left[\boldsymbol{\nabla}^2v_r-\frac{v_r}{r^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}\right]+b_r, \\
\frac{\partial v_\theta}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})v_\theta+\frac{v_rv_\theta}{r} &= -\frac{1}{\rho r}\frac{\partial p}{\partial\theta}+\nu\left[\boldsymbol{\nabla}^2v_\theta-\frac{v_\theta}{r^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial\theta}\right]+b_\theta, \\
\frac{\partial v_z}{\partial t}+(\boldsymbol{v}\cdot\boldsymbol{\nabla})v_z &= -\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\boldsymbol{\nabla}^2v_z+b_z.
\end{aligned}
$$

where $\nu=\mu/\rho$, and

$$
\begin{aligned}
\boldsymbol{v}\cdot\boldsymbol{\nabla} &\equiv v_r\frac{\partial}{\partial r}+\frac{v_\theta}{r}\frac{\partial}{\partial\theta}+v_z\frac{\partial}{\partial z} \\
\boldsymbol{\nabla}^2 &\equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}.
\end{aligned}
$$