Indian Institute of Science, Bangalore

ME 243: Endsemester Exam

Date: 12/12/12.
Duration: 9.30 a.m.–12.30 p.m.
Maximum Marks: 100

Instructions:
You may directly use the formulae at the back.

1. We saw in the test that $\{\boldsymbol{I}, \boldsymbol{W}, \boldsymbol{W}^2\}$ is linearly independent. Is $\{\boldsymbol{I}, \boldsymbol{Q}, \boldsymbol{Q}^T\}$ linearly dependent or linearly independent? If not linearly independent, under what conditions is it linearly dependent? (15)

2. Assuming the relation $\boldsymbol{x}^* = \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{c}$ between the position vectors $\boldsymbol{x}$ and $\boldsymbol{x}^*$ in two frames of reference, find the relation between $\boldsymbol{\nabla}_{\boldsymbol{x}^*} \cdot \boldsymbol{v}^*$ and $\boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \boldsymbol{v}$. Starting from the *integral form* of the balance of angular momentum, *derive* the consequences of assuming 'frame indifference' of this integral form (balance law has the same form in all frames of reference). *Do not* assume balance of mass or linear momentum to hold. You may assume that $dV^* = dV$, and (35)

$$\rho^*(\boldsymbol{x}^*, t) = \rho(\boldsymbol{x}, t), \quad \boldsymbol{\tau}^T = \boldsymbol{\tau}, \quad \boldsymbol{t} = \boldsymbol{\tau}\boldsymbol{n},$$
$$\boldsymbol{t}^*(\boldsymbol{x}^*, t, \boldsymbol{n}^*) = \boldsymbol{Q}\boldsymbol{t}(\boldsymbol{x}, t, \boldsymbol{n}), \quad (\boldsymbol{b}^* - \boldsymbol{a}^*) = \boldsymbol{Q}(\boldsymbol{b} - \boldsymbol{a}),$$
$$\boldsymbol{\tau}^* = \boldsymbol{Q}\boldsymbol{\tau}\boldsymbol{Q}^T, \quad \boldsymbol{\nabla}_{\boldsymbol{x}^*} \cdot \boldsymbol{\tau}^* = \boldsymbol{Q}\boldsymbol{\nabla}_{\boldsymbol{x}} \cdot \boldsymbol{\tau}.$$

(Hint: Careful with the use of transport theorems.)

3. Let x^*-y^* be a 'fixed' frame of reference with body force $\boldsymbol{b} = g\boldsymbol{e}_x^*$. A rod of length l and mass per unit length m, which is pinned at one end, oscillates in the x^*-y^* plane as shown in Fig. 1. The pin support itself oscillates along the y^* axis with its distance from the x^*-y^* origin being given by $u = u_0 \sin \omega t$. Using the appropriate transformation laws, find the body force field in the x-y frame. Approximating the rod as a one-dimensional rigid body ($\rho dV \equiv m\, dx$), and applying the appropriate balance laws in integral form in the x-y frame, find the governing equation for the angle θ, and the reactions exerted at the pinned-support with respect to the x-y axes. Linearize the governing equation for θ and solve it. (25)

4. We have seen that the equation of rigid motion in a Cartesian system is given by $\boldsymbol{v} = \boldsymbol{W}\boldsymbol{x} + \boldsymbol{c}$, where $\boldsymbol{W} \in$ Skw. By using appropriate tensorial transformations, find the expressions for (v_r, v_θ, v_z) for a rigid motion. Your final expressions for these components should be functions of (r, θ, z) (including some constants). Now, solve the following problem. A Newtonian, incompressible viscous fluid is contained in a rotating cylindrical container of radius a rotating with constant angular speed ω_0 as shown in Fig. 2. The (25)

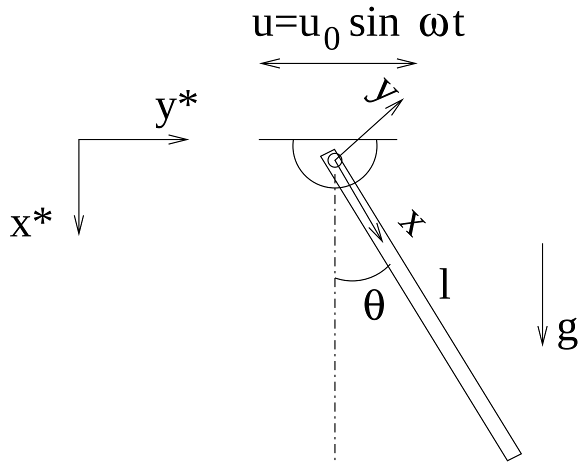

Figure 1: Oscillating rod.

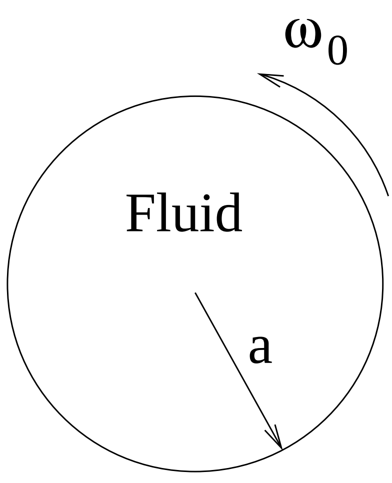

Figure 2: Problem 4.

fluid inside reaches to a steady-state velocity. Assuming $v_z = 0$, and assuming the velocity field in a *stationary* frame of reference with $\boldsymbol{b} = \boldsymbol{0}$ to be of the form you have derived above, first find the velocity field and then the pressure field from the momentum equations assuming $p|_{r=0} = 0$.

Some relevant formulae

$$\frac{d}{dt}\int_{V(t)} f(\boldsymbol{x},t)\,dV = \int_{V(t)} \frac{\partial f}{\partial t}\,dV + \int_{S(t)} f(\boldsymbol{v}\cdot\boldsymbol{n})\,dS,$$

$$\frac{d}{dt}\int_{V(t)} \rho f(\boldsymbol{x},t)\,dV = \int_{V(t)} \rho\frac{Df}{Dt}\,dV = \int_{V(t)} \rho\left\{\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot(\boldsymbol{\nabla} f)\right\}dV,$$

$$\int_{V(t)} (\boldsymbol{x}\times\boldsymbol{W}\boldsymbol{x})\phi\,dV = 0 \quad \forall\boldsymbol{W}\in\mathrm{Skw}\ \&\ \forall V(t) \implies \phi = 0,$$

$$\boldsymbol{W} = |\boldsymbol{w}|\,(\boldsymbol{r}\otimes\boldsymbol{q} - \boldsymbol{q}\otimes\boldsymbol{r}),$$

$$\boldsymbol{Q} = \boldsymbol{I} + \sin\alpha\,\boldsymbol{W} + (1-\cos\alpha)\boldsymbol{W}^2,$$

$$\boldsymbol{b} = \boldsymbol{Q}^T\left[\boldsymbol{b}^* - \ddot{\mathbf{c}}\right] - \dot{\boldsymbol{\Omega}}\times\boldsymbol{x} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{x}) - 2\boldsymbol{\Omega}\times\boldsymbol{v}.$$

$$Q_{ij} = \boldsymbol{e}_i^*\cdot\boldsymbol{e}_j,$$

$$\boldsymbol{\Omega} = \begin{bmatrix}\dot{\boldsymbol{e}}_2\cdot\boldsymbol{e}_3\\ \dot{\boldsymbol{e}}_3\cdot\boldsymbol{e}_1\\ \dot{\boldsymbol{e}}_1\cdot\boldsymbol{e}_2\end{bmatrix},$$

$$Q_{ij} = \bar{\boldsymbol{e}}_i\cdot\boldsymbol{e}_j.$$

$$D_{rr} = \frac{\partial u_r}{\partial r}, \qquad D_{r\theta} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial u_r}{\partial\theta} + r\frac{\partial}{\partial r}\left(\frac{u_\theta}{r}\right)\right],$$

$$D_{\theta\theta} = \frac{1}{r}\left(\frac{\partial u_\theta}{\partial\theta} + u_r\right), \qquad D_{\theta z} = \frac{1}{2}\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial\theta}\right),$$

$$D_{zz} = \frac{\partial u_z}{\partial z}, \qquad D_{rz} = \frac{1}{2}\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right).$$

<u>The momentum equations in r, θ and z directions:</u>

$$\frac{\partial v_r}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})v_r - \frac{v_\theta^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\boldsymbol{\nabla}^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial\theta}\right] + b_r,$$

$$\frac{\partial v_\theta}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})v_\theta + \frac{v_r v_\theta}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} + \nu\left[\boldsymbol{\nabla}^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial\theta}\right] + b_\theta,$$

$$\frac{\partial v_z}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})v_z = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\boldsymbol{\nabla}^2 v_z + b_z.$$

where $\nu = \mu/\rho$, and

$$\boldsymbol{v}\cdot\boldsymbol{\nabla} \equiv v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial\theta} + v_z\frac{\partial}{\partial z}$$

$$\boldsymbol{\nabla}^2 \equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}.$$