Indian Institute of Science, Bangalore

ME 243: Endsemester Exam

Date: 12/12/14.
Duration: 9.00 a.m.–12.00 noon
Maximum Marks: 100

Instructions:
You may directly use the formulae at the back.

1. Assume that the heat flux is a function of the deformation gradient $\boldsymbol{F}$ and the temperature gradient $\boldsymbol{g} := \boldsymbol{\nabla}_x\theta$, i.e., $\boldsymbol{q} = \tilde{\boldsymbol{q}}(\boldsymbol{F}, \boldsymbol{g})$. Derive the relations for $\boldsymbol{F}^*$ and $\boldsymbol{g}^*$. Then state the conditions for $\tilde{\boldsymbol{q}}$ to be frame-indifferent and isotropic. (*Do not* try to find a characterization for a frame-indifferent, isotropic $\tilde{\boldsymbol{q}}$). Next, if $\boldsymbol{q}_0 := (\textbf{cof}\,\boldsymbol{F})^T\boldsymbol{q}$ and $\boldsymbol{g}_0 := \boldsymbol{\nabla}_X\tilde{\theta}$ is the referential temperature gradient, find the conditions for $\tilde{\boldsymbol{q}}_0(\boldsymbol{F}, \boldsymbol{g}_0)$ to be frame-indifferent and isotropic. Now evaluate if the following constitutive relations for the flux are frame-indifferent and/or isotropic or neither ($\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F}$, $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$, β_0, β_1, β_2 are constants, and $\mathbf{C}$ is a constant fourth-order tensor). (25)
(1) $\tilde{\boldsymbol{q}} = (\beta_0\boldsymbol{I} + \beta_1\boldsymbol{B} + \beta_2\boldsymbol{B}^2)\boldsymbol{g}$, (6) $\tilde{\boldsymbol{q}}_0 = (\beta_0\boldsymbol{I} + \beta_1\boldsymbol{B} + \beta_2\boldsymbol{B}^2)\boldsymbol{g}_0$.
(2) $\tilde{\boldsymbol{q}} = (\beta_0\boldsymbol{I} + \beta_1\boldsymbol{C} + \beta_2\boldsymbol{C}^2)\boldsymbol{g}$, (7) $\tilde{\boldsymbol{q}}_0 = (\beta_0\boldsymbol{I} + \beta_1\boldsymbol{C} + \beta_2\boldsymbol{C}^2)\boldsymbol{g}_0$.
(3) $\tilde{\boldsymbol{q}} = (\mathbf{C}\boldsymbol{B})\boldsymbol{g}$. (8) $\tilde{\boldsymbol{q}}_0 = (\mathbf{C}\boldsymbol{B})\boldsymbol{g}_0$.
(4) $\tilde{\boldsymbol{q}} = (\mathbf{C}\boldsymbol{C})\boldsymbol{g}$. (9) $\tilde{\boldsymbol{q}}_0 = (\mathbf{C}\boldsymbol{C})\boldsymbol{g}_0$.
(5) $\tilde{\boldsymbol{q}} = \beta_0(\boldsymbol{g}\cdot\boldsymbol{g})\boldsymbol{g}$. (10) $\tilde{\boldsymbol{q}}_0 = \beta_0(\boldsymbol{g}_0\cdot\boldsymbol{g}_0)\boldsymbol{g}_0$.
(Hint: Isotropy can be considered as a rotation of the reference configuration $\boldsymbol{X} = \boldsymbol{Q}\bar{\boldsymbol{X}}$ leading to $\bar{\boldsymbol{F}} = \boldsymbol{F}\boldsymbol{Q}$. Assume that $\boldsymbol{g} = \boldsymbol{\nabla}_x\theta$ remains unaffected by this rotation.)

2. Let $\{\lambda_1, \lambda_2, \lambda_3\}$ be the distinct (not necessarily nonzero) eigenvalues of a tensor $\boldsymbol{S} \in \text{Sym}$. (20)

   (a) Find the eigenvalues of $\boldsymbol{S} - \lambda_1\boldsymbol{I}$. Using $\textbf{cof}\,\boldsymbol{S} = I_2\boldsymbol{I} - (\operatorname{tr}\boldsymbol{S})\boldsymbol{S} + \boldsymbol{S}^2$, find $\textbf{cof}\,(\boldsymbol{S} - \lambda_1\boldsymbol{I})$.

   (b) Using the fact that λ_1 satisfies the characteristic equation of $\boldsymbol{S}$, i.e., $\det(\boldsymbol{S} - \lambda_1\boldsymbol{I}) = 0$, find $\partial\lambda_1/\partial\boldsymbol{S}$. Your expression should be an explicit function of $\boldsymbol{S}$ and the eigenvalues λ_i (try to reduce to the simplest possible form). By a permutation of indices, find $\partial\lambda_2/\partial\boldsymbol{S}$ and $\partial\lambda_3/\partial\boldsymbol{S}$.

   (c) Let $\{\lambda_1, \lambda_2, \lambda_3\}$ denote the eigenvalues of $\boldsymbol{V} := \sqrt{\boldsymbol{B}} = \sqrt{\boldsymbol{F}\boldsymbol{F}^T}$. If

   $$W(\boldsymbol{B}) = \mu\left[(\log\lambda_1)^2 + (\log\lambda_2)^2 + (\log\lambda_3)^2\right],$$

   where μ is a constant, find $\boldsymbol{S} = 2\partial W/\partial\boldsymbol{C}$.

3. Let X^*-Y^*-Z^* be a 'fixed' frame of reference with basis vectors ($\boldsymbol{e}_1^*$, $\boldsymbol{e}_2^*$, $\boldsymbol{e}_3^*$). Let X-Y-Z be a rotating frame of reference with basis vectors ($\boldsymbol{e}_1$, $\boldsymbol{e}_2$, $\boldsymbol{e}_3$) making an angle $\theta(t)$ with the fixed frame as shown in Fig. 1. Assume (25)

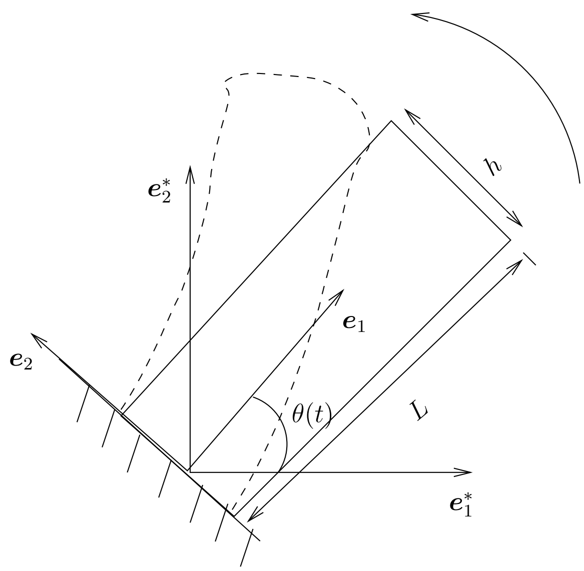

Figure 1: Rotating cantilever beam with the deformed shape shown by the dotted line.

$\boldsymbol{b}^* = \boldsymbol{0}$ and assume a St. Venant-Kirchhoff material with density $\rho_0(\boldsymbol{X})$. A cantilever beam whose fixed support is aligned with $\boldsymbol{e}_2$, and hence rotates with the same angular speed as the rotating frame, gets deformed (the deformed shape at some instant of time is shown by a dotted line; do *not* assume the cantilever to have reached 'steady-state' with respect to the rotating observer) from the original reference configuration (which is shown as a rectangle) due to this rotation. Note that this reference configuration is *fixed* at all times with respect to the rotating observer. Assuming that the beam occupies the rectangular domain at $t = 0$ and has zero initial velocity, write the complete problem formulation for the displacements with respect to this reference configuration (Lagrangian formulation). Your formulation should include the initial conditions and the in-plane boundary conditions; you may ignore the out-of-plane deformations. There should be as many equations as unknowns in your formulation (e.g., if you use $\boldsymbol{F}$, then define the inplane $\boldsymbol{X}$ and $\boldsymbol{u}$ that you are using with the help of a figure, and then express $\boldsymbol{F}$ in terms of these quantities). Do not attempt to solve any of these equations but derive the formulae that you use, say for $\boldsymbol{\Omega}$ or $\ddot{\boldsymbol{c}}$. Apart from the boundary conditions where components need to be used, the remaining equations can be in tensorial form.

Next, derive the linearized equations assuming that the displacement vector and its gradients are small with respect to the *(rotating) reference configuration.* (thus, with respect to the fixed observer, a 'small' deformation is superimposed on a 'large' rigid rotation).

4. (a) Taking the dot product of the linear momentum equation in Eulerian form with the velocity vector, and integrating over a material volume, derive a relation for the rate of kinetic energy. Derive any tensorial (30)

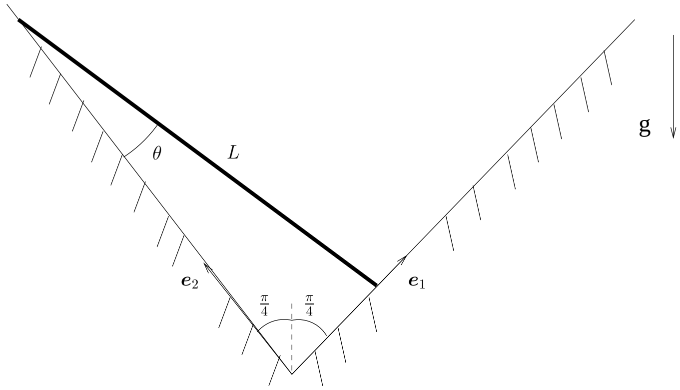

Figure 2: Sliding rod.

identities that you need on the way.

(b) A rigid bar of length L and mass per unit length $m = M/L$ is released from rest under gravity loading from an initial angle $\theta = \pi/4$ in the setup shown in Fig. 2. Assuming that the ends of the bar slide frictionlessly along the surface, find the angular velocity of the bar when $\theta = \pi/6$ (Hint: Is any quantity conserved?). Approximate the rod as a one-dimensional rigid body ($\rho dV \equiv m\, dx$). You may directly use $\boldsymbol{v} - \dot{\bar{\boldsymbol{x}}} = \boldsymbol{\omega} \times (\boldsymbol{x} - \bar{\boldsymbol{x}})$, where $\boldsymbol{x}$ is the position vector, $\bar{\boldsymbol{x}}(t)$ is the position of the centroid, and $\boldsymbol{\omega} = (0, 0, \dot{\theta})$. You may also directly use $\int_V \rho(\boldsymbol{x} - \bar{\boldsymbol{x}}) \cdot (\boldsymbol{x} - \bar{\boldsymbol{x}})\, dV = ML^2/12$. Derive any other relations that you may need. *Do not* use the Euler equations. Solve using first principles.

Some relevant formulae

$$\mathbf{cof}\,(\boldsymbol{AB}) = (\mathbf{cof}\,\boldsymbol{A})(\mathbf{cof}\,(\boldsymbol{B}),$$

$$\det(\boldsymbol{T} + \boldsymbol{U}) = \det \boldsymbol{T} + \mathbf{cof}\,\boldsymbol{T} : \boldsymbol{U} + \mathbf{cof}\,\boldsymbol{U} : \boldsymbol{T} + \det \boldsymbol{U},$$

$$\boldsymbol{\nabla}_X \cdot \boldsymbol{T} = J \boldsymbol{\nabla}_x \cdot \boldsymbol{\tau},$$

$$\rho_0 = \tilde{\rho} J,$$

$$\boldsymbol{t}_0 = \boldsymbol{T}\boldsymbol{n}_0,$$

$$\frac{d}{dt}\int_{V(t)} f(\boldsymbol{x}, t)\, dV = \int_{V(t)} \frac{\partial f}{\partial t}\, dV + \int_{S(t)} f(\boldsymbol{v} \cdot \boldsymbol{n})\, dS,$$

$$\frac{d}{dt}\int_{V(t)} \rho f(\boldsymbol{x}, t)\, dV = \int_{V(t)} \rho \frac{Df}{Dt}\, dV = \int_{V(t)} \rho \left\{ \frac{\partial f}{\partial t} + \boldsymbol{v} \cdot (\boldsymbol{\nabla} f) \right\} dV,$$

$$\boldsymbol{b} = \boldsymbol{Q}^T \left[\boldsymbol{b}^* - \ddot{\mathbf{c}}\right] - \dot{\boldsymbol{\Omega}} \times \boldsymbol{x} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}) - 2\boldsymbol{\Omega} \times \boldsymbol{v}.$$

$$Q_{ij} = \boldsymbol{e}_i^* \cdot \boldsymbol{e}_j,$$

$$\boldsymbol{\Omega} = \begin{bmatrix} \dot{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_3 \\ \dot{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_1 \\ \dot{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_2 \end{bmatrix},$$