Indian Institute of Science, Bangalore

ME 243: Endsemester Exam

Date: 11/12/15.
Duration: 9.00 a.m.–12.00 noon
Maximum Marks: 100

Instructions:
You may directly use the formulae at the back.

1. Let $\boldsymbol{Q}_1, \boldsymbol{Q} \in \text{Orth}^+$. Then show that $\boldsymbol{Q}_1 = \boldsymbol{Q}_2$ if and only if (25)

$$\boldsymbol{e}(\boldsymbol{Q}_1) = \boldsymbol{e}(\boldsymbol{Q}_2) \text{ and } \operatorname{tr}\boldsymbol{Q}_1 = \operatorname{tr}\boldsymbol{Q}_2 \text{ and } \hat{\boldsymbol{a}}(\boldsymbol{Q}_1 - \boldsymbol{Q}_1^T)\cdot\boldsymbol{e}(\boldsymbol{Q}_1) = \hat{\boldsymbol{a}}(\boldsymbol{Q}_2 - \boldsymbol{Q}_2^T)\cdot\boldsymbol{e}(\boldsymbol{Q}_2),$$

where $\boldsymbol{e}(\boldsymbol{Q})$ denotes the axis of $\boldsymbol{Q} \in \text{Orth}^+$, i.e., the eigenvector corresponding to the eigenvalue 1 of $\boldsymbol{Q}$, and $\hat{\boldsymbol{a}}(\boldsymbol{W})$ denotes the axial vector of any $\boldsymbol{W} \in \text{Skw}$. You may directly use the following representation for $\boldsymbol{Q}$ in order to prove this result:

$$\boldsymbol{Q} = e^{\boldsymbol{W}} = \boldsymbol{I} + \frac{\sin(|\boldsymbol{w}|)}{|\boldsymbol{w}|}\boldsymbol{W} + \frac{[1 - \cos(|\boldsymbol{w}|)]}{|\boldsymbol{w}|^2}\boldsymbol{W}^2,$$

where $\boldsymbol{w} = \hat{\boldsymbol{a}}(\boldsymbol{W})$. Assume $\sin(|\boldsymbol{w}_1|)\boldsymbol{W}_1 \neq \boldsymbol{0}$ and $\sin(|\boldsymbol{w}_2|)\boldsymbol{W}_2 \neq \boldsymbol{0}$, so that $\boldsymbol{Q}_1 = e^{\boldsymbol{W}_1}$ and $\boldsymbol{Q}_2 = e^{\boldsymbol{W}_2}$ are unsymmetric.

2. Evaluate if the following constitutive relations are frame-indifferent: (25)

$$\begin{aligned} \boldsymbol{S} &= \lambda(\operatorname{tr}\boldsymbol{V})\boldsymbol{I} + 2\mu\boldsymbol{V}, \\ \dot{\boldsymbol{\tau}} - \boldsymbol{\tau}\boldsymbol{L}^T - \boldsymbol{L}\boldsymbol{\tau} &= \mu(\boldsymbol{\tau}\boldsymbol{D} + \boldsymbol{D}\boldsymbol{\tau}), \\ \dot{\boldsymbol{q}} - \dot{\boldsymbol{R}}\boldsymbol{R}^T\boldsymbol{q} &= \mu\boldsymbol{B}\boldsymbol{g}, \end{aligned}$$

where a superposed dot represents the material time derivative, $\boldsymbol{V}$ and $\boldsymbol{R}$ are the stretch and rotation tensors in the polar decomposition of $\boldsymbol{F}$, $\boldsymbol{q}$ and $\boldsymbol{g}$ are the (spatial) heat flux and temperature gradient, λ and μ are constants, $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$, $\boldsymbol{L}$ is the velocity gradient and $\boldsymbol{S}$ is the second Piola-Kirchhoff stress. You may directly use $\boldsymbol{x}^* = \boldsymbol{Q}(t)\boldsymbol{x} + \boldsymbol{c}(t)$ and $\boldsymbol{F}^* = \boldsymbol{Q}(t)\boldsymbol{F}$, but derive the transformation relations for other kinematical quantities. State any assumptions you make clearly.

3. A rigid rectangular block of mass M slides along a frictionless rigid groove in a rotating disc as shown in Fig. 1. The center of mass of the block is at a distance x_0 from the center of the disc, and the block is at rest at $t = 0$. The groove makes an angle $\theta(t)$ with the horizontal. Recall that the center of mass is given by $M\bar{\boldsymbol{x}} = \int_{V(t)} \rho\boldsymbol{x}\, dV$. Using the appropriate balance laws in integral form (for a material volume), and assuming that the center of mass of the block always moves along the centerline of the groove, derive the (25)

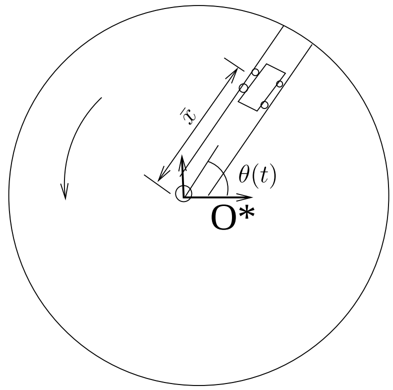

Figure 1: Rigid block sliding along a rigid groove in a rotating disc.

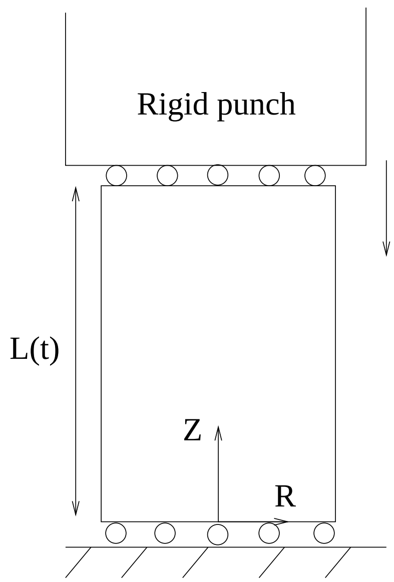

Figure 2: A cylindrical bar squeezed between two rigid frictionless surfaces.

governing equation for the distance of the center of the mass from the center of the disc $\bar{x}(t)$, and state the initial conditions for solving this equation (Do *not* attempt to solve this governing equation.). Next, find the total normal force (along with its direction) exerted by the groove on the block as a function of $\bar{x}(t)$ and $\theta(t)$ (recall that the tangential force is zero since all surfaces are assumed to be frictionless). You may neglect gravity and treat the problem as two-dimensional.

4. A circular cylinder made of an incompressible material with initial radius R_0, length L_0 and density ρ_0 in its natural state is squeezed between a moving punch and a rigid bottom surface as shown in Fig. 2. Assume that the top and bottom surfaces are frictionless, and that the bar remains prismatic, i.e., the radius of any cross section along Z is the same. Neglect body forces, and assume the constitutive relation to be given by $\boldsymbol{\tau} = -p\boldsymbol{I} + \beta_1 \boldsymbol{B}$, where β_1 (25)

is a constant. The motion of the punch is controlled so that the length of the cylinder at time t is given by the prescribed function $L(t) = L_0(1 - \alpha t)$ where α is a positive constant. After making suitable assumptions about the nature of the deformation (state these assumptions clearly), formulate the deformation map between (r, θ, z) and (R, Θ, Z) in terms of the known quantities such as $L(t)$, L_0 etc. Using the relation

$$F_{iJ} = \frac{h_i}{h_J} \frac{\partial \hat{\chi}_i}{\partial \eta_J}, \quad \text{no sum on } i,\ J.$$

with $h_i \equiv (1, r, 1)$, $h_J \equiv (1, R, 1)$, and $\eta_J \equiv (R, \Theta, Z)$, find the deformation gradient. Using the governing equations and boundary conditions in the *deformed* configuration find the pressure field. State the expression for the total normal force exerted by the punch on the top surface in terms of p (no need to evaluate any integrals that arise).

Some relevant formulae

$$w_i = -\frac{1}{2}\epsilon_{ijk} W_{jk},$$
$$W_{ij} = -\epsilon_{ijk} w_k,$$
$$\boldsymbol{b} = \boldsymbol{Q}^T \left[\boldsymbol{b}^* - \ddot{\mathbf{c}}\right] - \dot{\boldsymbol{\Omega}} \times \boldsymbol{x} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}) - 2\boldsymbol{\Omega} \times \boldsymbol{v}.$$
$$Q_{ij} = \boldsymbol{e}_i^* \cdot \boldsymbol{e}_j,$$
$$\boldsymbol{\Omega} = \begin{bmatrix} \dot{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_3 \\ \dot{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_1 \\ \dot{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_2 \end{bmatrix},$$

If $\boldsymbol{T} \in \text{Lin}$, the components of $\boldsymbol{\nabla} \cdot \boldsymbol{T}$ in the cylindrical coordinate system are

$$(\boldsymbol{\nabla} \cdot \boldsymbol{T})_r = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{T_{rr} - T_{\theta\theta}}{r},$$
$$(\boldsymbol{\nabla} \cdot \boldsymbol{T})_\theta = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{T_{r\theta} + T_{\theta r}}{r},$$
$$(\boldsymbol{\nabla} \cdot \boldsymbol{T})_z = \frac{\partial T_{zr}}{\partial r} + \frac{1}{r}\frac{\partial T_{z\theta}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{zr}}{r}.$$