Indian Institute of Science, Bangalore

ME 243: Endsemester Exam

Date: 14/12/16.
Duration: 9.30 a.m.–12.30 p.m.
Maximum Marks: 100

Instructions:

You may directly use the formulae at the back.

1. We saw that for an isotropic material, the Cauchy stress is given by (25)

$$\frac{J}{2}\hat{\boldsymbol{\tau}} = \frac{\partial \hat{W}}{\partial I_1}\boldsymbol{B} + \frac{\partial \hat{W}}{\partial I_2}(I_1\boldsymbol{B} - \boldsymbol{B}^2) + \frac{\partial \hat{W}}{\partial I_3}I_3\boldsymbol{I},$$

where I_i, $i = 1, 2, 3$ are the principal invariants of $\boldsymbol{B}$, and $W = \hat{W}(I_1, I_2, I_3)$.

(a) Using the above relation, find a relation between $J\hat{\boldsymbol{\tau}}/2$ and $\boldsymbol{B}\frac{\partial \hat{W}}{\partial \boldsymbol{B}}$.

(b) Using

$$\frac{\partial \hat{W}}{\partial \boldsymbol{V}} : \boldsymbol{U} = D\hat{W}(\boldsymbol{V})[\boldsymbol{U}],$$

and the chain rule, find a relation between

$$\boldsymbol{V}\frac{\partial \hat{W}}{\partial \boldsymbol{V}} \text{ and } \boldsymbol{B}\frac{\partial \hat{W}}{\partial \boldsymbol{B}}$$

when the material is isotropic, i.e., when $\partial\hat{W}/\partial\boldsymbol{B}$ is a polynomial in $\boldsymbol{B}$.

(c) Combine the above two parts and find a relation between

$$J\hat{\boldsymbol{\tau}} \text{ and } \boldsymbol{V}\frac{\partial \hat{W}}{\partial \boldsymbol{V}}$$

(d) Given that $\hat{W} = 2\mu\,[\det \boldsymbol{V}(\operatorname{tr}\boldsymbol{V} - 4) + 1]$, where μ is a constant, use the above relation that you have derived to find an expression for $\hat{\boldsymbol{\tau}}$ in terms of $\boldsymbol{V}$.

(e) Directly verify if the constitutive relation that you have obtained for the Cauchy stress is frame-indifferent and isotropic (you may need to derive a relation between $\boldsymbol{V}^*$ and $\boldsymbol{V}$ etc. to do this).

2. Given that (15)

$$\boldsymbol{\nabla}_X \cdot \boldsymbol{T} + \rho_0\boldsymbol{b}_0 = \rho_0\tilde{\boldsymbol{a}},$$

has the same form in all frames of reference, find a relation between $\boldsymbol{b}_0^*$ and $\boldsymbol{b}_0$ (you may express this relationship in terms of $\boldsymbol{Q}$, $\dot{\boldsymbol{Q}}$ etc. without converting to $\boldsymbol{\Omega}$). Using $\boldsymbol{t}_0 = |(\mathbf{cof}\,\boldsymbol{F})\boldsymbol{n}_0|\,\boldsymbol{t}$, also find a relation between $\boldsymbol{t}_0^*$ and $\boldsymbol{t}_0$. You may assume $\boldsymbol{t}^* = \boldsymbol{Q}(t)\boldsymbol{t}$, $\boldsymbol{\tau}^* = \boldsymbol{Q}(t)\boldsymbol{\tau}\boldsymbol{Q}^T(t)$ and $\boldsymbol{x}^* = \boldsymbol{Q}(t)\boldsymbol{x} + \boldsymbol{c}$, but derive any other relations that are required. Work totally within the Lagrangian framework for the kinematical variables.

3. A hollow sphere of inner radius a and outer radius b is subjected to a uniform pressure load p at its inner surface $r = a$ as shown in Fig. 1. The outer surface $r = b$ is in contact with a smooth rigid surface so that it cannot move radially outward. The constitutive relation is given by $\boldsymbol{\tau} = \boldsymbol{F} - \boldsymbol{I}$ (of course this is not a valid constitutive relation, but assumed just to make you calculations simpler). Using appropriate symmetry arguments and boundary conditions, find the mapping $(r, \theta, \phi) = \boldsymbol{\chi}(R, \Theta, \Phi)$ assuming a static deformation and neglecting body forces (Hint: Make an intelligent guess based on symmetry arguments). You may simply state the governing equations and boundary conditions for the unknown functions that you assume. *Do not* attempt to solve these differential equations. (25)

4. Let x^*-y^* be a 'fixed' frame of reference with body force $\boldsymbol{b}^* = g\boldsymbol{e}^*_x$. A rod of length l and mass per unit length m, which is pinned at one end, oscillates in the x^*-y^* plane as shown in Fig. 2. A rigid block of mass M slides *frictionlessly* along the pendulum as shown. You may treat the sliding mass M as being lumped at $\bar{x}(t)$ for the purpose of the derivation. Using the appropriate transformation laws, find the body force field in the x-y frame. Approximating the rod as a one-dimensional rigid body ($\boldsymbol{x} = x\boldsymbol{e}_x$, $\rho dV \equiv m\,dx$), and applying the appropriate balance laws in integral form in the x-y frame, find the governing equations for the position of the mass M (i.e., $\bar{x}(t)$) and the angle made by the pendulum $\theta(t)$. Do not attempt to solve these equations. Do not use the Euler equations or any approach other than the one indicated. (35)

Some relevant formulae

$$\frac{\partial I_1}{\partial \boldsymbol{T}} = \boldsymbol{I},$$

$$\frac{\partial I_2}{\partial \boldsymbol{T}} = (\operatorname{tr}\boldsymbol{T})\boldsymbol{I} - \boldsymbol{T}^T,$$

$$\frac{\partial I_3}{\partial \boldsymbol{T}} = \mathbf{cof}\,\boldsymbol{T},$$

$$w_i = -\frac{1}{2}\epsilon_{ijk}W_{jk},$$

$$W_{ij} = -\epsilon_{ijk}w_k,$$

$$\boldsymbol{b} = \boldsymbol{Q}^T\left[\boldsymbol{b}^* - \ddot{\mathbf{c}}\right] - \dot{\boldsymbol{\Omega}} \times \boldsymbol{x} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}) - 2\boldsymbol{\Omega} \times \boldsymbol{v}.$$

$$Q_{ij} = \boldsymbol{e}^*_i \cdot \boldsymbol{e}_j,$$

$$\boldsymbol{\Omega} = \begin{bmatrix} \dot{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_3 \\ \dot{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_1 \\ \dot{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_2 \end{bmatrix},$$

$$F_{iJ} = \frac{h_i}{h_J}\frac{\partial \hat{\chi}_i}{\partial \eta_J}, \quad \text{no sum on } i,\ J, \quad h_i \equiv (1, r, r\sin\theta), \quad h_J \equiv (1, R, R\sin\Theta).$$

The components of $\boldsymbol{\nabla} \cdot \boldsymbol{\tau}$ in the spherical coordinate system are

$$
\begin{aligned}
(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_r &= \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{r\phi}}{\partial \phi} + \frac{1}{r}\left(2\tau_{rr} - \tau_{\theta\theta} - \tau_{\phi\phi} + \cot\theta\, \tau_{r\theta}\right), \\
(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_\theta &= \frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{\theta\phi}}{\partial \phi} + \frac{1}{r}\left(\tau_{r\theta} + 2\tau_{\theta r} + \cot\theta(\tau_{\theta\theta} - \tau_{\phi\phi})\right), \\
(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_\phi &= \frac{\partial \tau_{\phi r}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\phi\theta}}{\partial \theta} + \frac{1}{r\sin\theta}\frac{\partial \tau_{\phi\phi}}{\partial \phi} + \frac{1}{r}\left(\tau_{r\phi} + 2\tau_{\phi r} + \cot\theta(\tau_{\theta\phi} + \tau_{\phi\theta})\right).
\end{aligned}
$$

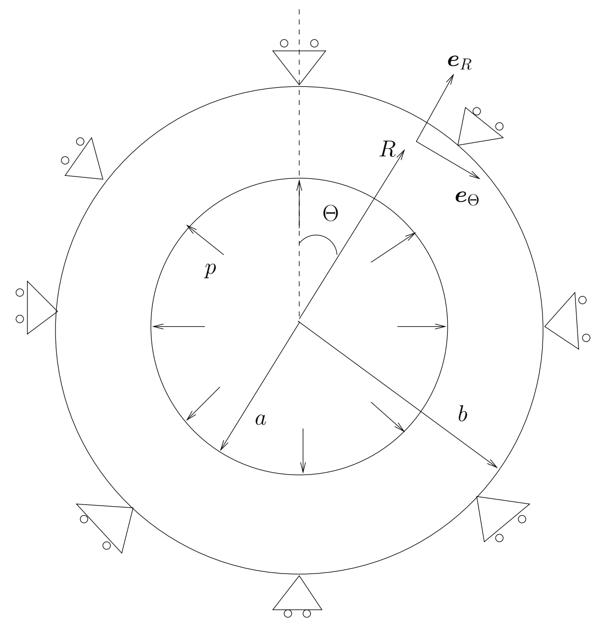

Figure 1: Hollow sphere subjected to internal pressure p.

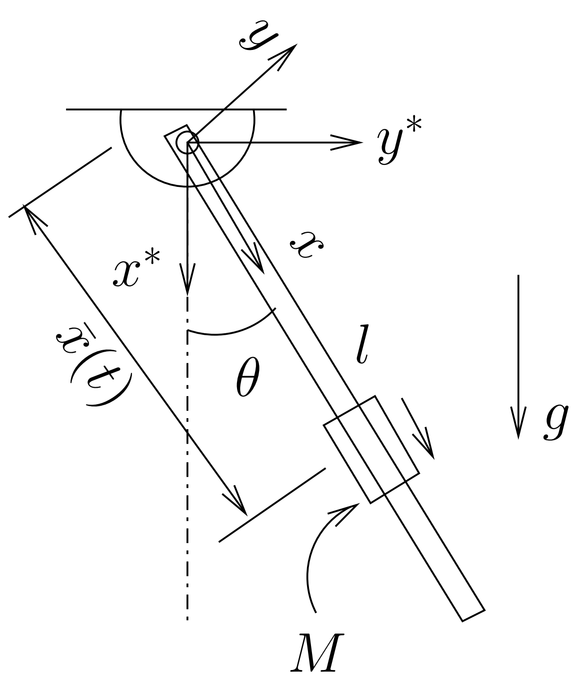

Figure 2: Pendulum with a sliding mass on it.