Indian Institute of Science, Bangalore

ME 243: Endsemester Exam

Date: 6/12/19.
Duration: 9.30 a.m.–12.30 p.m.
Maximum Marks: 100

Instructions:
You may directly use the formulae at the back.

1. Let $\boldsymbol{B}$ and $\boldsymbol{C}$ denote the left and right Cauchy Green strain tensors. The purpose of this exercise is to determine under what conditions the eigenvectors of $\boldsymbol{B}$ and those of the rate of deformation tensor $\boldsymbol{D}$ coincide. Let $\boldsymbol{B} = \sum_{i=1}^{3} \lambda_i^2 \boldsymbol{n}_i \otimes \boldsymbol{n}_i$, $\boldsymbol{C} = \sum_{i=1}^{3} \gamma_i^2 \boldsymbol{N}_i \otimes \boldsymbol{N}_i$. Assume λ_i and γ_i to be distinct and greater than zero throughout this exercise. (25)
   (a) If a relation exists between λ_i and γ_i, find this relation.
   (b) Find an expression for $\boldsymbol{D}$ in terms of $\dot{\lambda}_i$, λ_i, $\boldsymbol{n}_i$, $\boldsymbol{N}_i$ and $\dot{\boldsymbol{N}}_i$, where the superposed dot denotes a material derivative.
   (c) From this expression, find the necessary and sufficient conditions for the eigenvectors of $\boldsymbol{D}$ to be the same as the eigenvectors $\boldsymbol{n}_i$ of $\boldsymbol{B}$.

2. This is the nonlinear version of the shrink fit problem. A hollow circular disc of initial inner and outer radii b and c is fitted over a solid circular cylinder of initial radius a where $a > b$ (This is generally done by heating the hollow circular disc, slipping it over the solid cylinder and then cooling the entire assembly to room temperature; solve this problem under the static framework within the purely mechanical context without considering any thermal effects). The constitutive equation is given by $\boldsymbol{\tau} = \mu \boldsymbol{B}$, where μ is a constant. By assuming an appropriate two-dimensional deformation field (i.e., $z = Z$) in the cylinder and hollow disc, find the governing equations for the unknown functions. State the appropriate boundary and interface conditions (which have to be stated precisely without using any assumptions such as those made in linearized elasticity). *Do not* attempt to solve the governing equations that result for your unknown fields. (20)

3. By taking the dot product of the linear momentum equation with the velocity $\boldsymbol{v}$ and integrating over the material volume, derive the integral form of the mechanical energy balance equation. Consider a rigid rod sliding against a frictionless wall and floor under the action of gravity as shown in Fig. 1. For the conditions given in this problem, derive from the mechanical energy balance equation you have derived, a conservation law of the form (25)

$$\int_{V(t)} (.) \rho \, dV = \text{constant}. \tag{1}$$

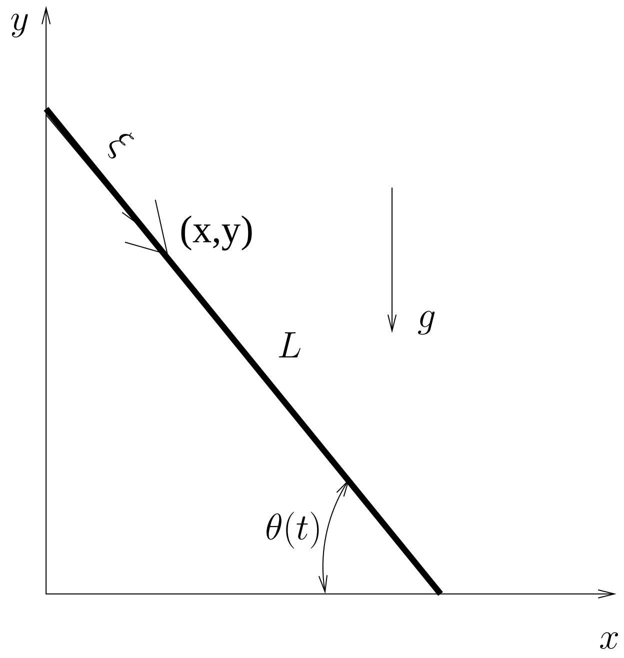

Figure 1: Problem 3.

Now use Eqn. (1) to derive the governing equation for $\dot{\theta}$, assuming that the rod is at rest at the initial position $\theta = \theta_0$. State the initial condition for solving this equation. *Do not* attempt to solve this governing differential equation. For the purpose of carrying out the integrations in Eqn. (1), assume the rod to be one-dimensional, express the coordinates of a typical point (x, y) on the rod in terms the coordinate ξ which runs along the length as shown in the figure, and assume $\rho\, dV = m\, d\xi$, where m, the mass per unit length, is a constant.

Next use the appropriate balance law (involving the center of mass) to find the reaction forces at the ends of the rod as functions of $\theta(t)$ and its derivatives (which are assumed to be known from the previous part), g, m and L.

4. The constitutive relations for a homogeneous solid are given by (30)

$$\psi = \tilde{\psi}(\boldsymbol{F}, \boldsymbol{F}^T \boldsymbol{\nabla}_X \tilde{\boldsymbol{v}}),$$
$$\boldsymbol{S} = \tilde{\boldsymbol{S}}(\boldsymbol{F}, \boldsymbol{F}^T \boldsymbol{\nabla}_X \tilde{\boldsymbol{v}}),$$

where $\tilde{\boldsymbol{v}}$ is the Lagrangian velocity vector.

(a) Using the principle of material frame-indifference (MFI) and the given constitutive relations, *derive* the list of kinematical variables on which ψ and $\boldsymbol{S}$ should depend, so that MFI is automatically satisfied. You may directly use $\boldsymbol{F}^* = \boldsymbol{QF}$. Derive the required formulae for other kinematical variables. (Hint: Try to reduce the list to Lagrangian variables and their material derivatives).

(b) Using the isothermal version of the Clausius-Duhem inequality given by

$$\left(\frac{\partial \tilde{\psi}}{\partial t}\right)_X - \frac{1}{\rho_0}\boldsymbol{T} : \dot{\boldsymbol{F}} \leq 0,$$

where $\boldsymbol{T} = \boldsymbol{F}\boldsymbol{S}$ is the first Piola-Kirchhoff stress tensor, find the equalities and inequalities (necessary and sufficient conditions) that result. In the proof for this part, *do not* attempt to construct an admissible thermodynamic process; assume that such a process can be constructed for arbitrary values of your kinematical variables (Hint: Split the stress $\boldsymbol{S}$ into an equilbrium part (defined as the stress when an appropriate kinematical variable is set to zero), and a nonequilibrium part). *Justify* all steps.

Some relevant formulae

$$\boldsymbol{R} : (\boldsymbol{S}\boldsymbol{T}) = (\boldsymbol{S}^T\boldsymbol{R}) : \boldsymbol{T} = (\boldsymbol{R}\boldsymbol{T}^T) : \boldsymbol{S} = (\boldsymbol{T}\boldsymbol{R}^T) : \boldsymbol{S}^T,$$

$$F_{iJ} = \frac{h_i}{h_J}\frac{\partial \hat{\chi}_i}{\partial \eta_J}, \quad \text{no sum on } i,\ J, \quad h_i \equiv (1, r, 1), \quad h_J \equiv (1, R, 1).$$

If $\boldsymbol{\tau}$ is symmetric tensor-valued field, then the components of $\boldsymbol{\nabla}_x \cdot \boldsymbol{\tau}$ with respect to a cylindrical coordinate system are

$$(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_r = \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r},$$
$$(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_\theta = \frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r}.$$