Indian Institute of Science, Bangalore

ME 243: Endsemester Exam

Date: 4/12/00.
Duration: 2.30 p.m.–5.30 p.m.
Maximum Marks: 200

1. The purpose of this exercise is to show that the axis of $\boldsymbol{Q} \in \mathrm{Orth}^+ - \{\boldsymbol{I}\}$ is a one-dimensional subspace. Towards this end (30)

   - Find all three principal invariants in terms of constants or $\operatorname{tr}\boldsymbol{Q}$ (use appropriate formulae to achieve this).
   - Solve the characteristic equation to find the eigenvalues of $\boldsymbol{Q}$ as a function of $\operatorname{tr}\boldsymbol{Q}$.
   - Use the Euler angle representation of $\boldsymbol{Q}$, i.e.,

   $$\boldsymbol{Q} = \begin{bmatrix} \cos\phi\cos\psi - \cos\theta\sin\psi\sin\phi & -\sin\phi\cos\psi - \cos\theta\sin\psi\cos\phi & \sin\theta\sin\psi \\ \cos\phi\sin\psi + \cos\theta\cos\psi\sin\phi & -\sin\phi\sin\psi + \cos\theta\cos\psi\cos\phi & -\sin\theta\cos\psi \\ \sin\phi\sin\theta & \cos\phi\sin\theta & \cos\theta \end{bmatrix}.$$

   to find an expression for $\operatorname{tr}\boldsymbol{Q}$, and use that to deduce that the axis of $\boldsymbol{Q}$ is a one-dimensional subspace.

2. Starting from the relation (50)

   $$J = \epsilon_{ijk}\frac{\partial\chi_1}{\partial X_i}\frac{\partial\chi_2}{\partial X_j}\frac{\partial\chi_3}{\partial X_k}.$$

   show that $DJ/Dt = J(\boldsymbol{\nabla}\cdot\boldsymbol{v})$. (Hint: Use the relation $\epsilon_{pqr}(\det\boldsymbol{T}) = \epsilon_{ijk}T_{pi}T_{qj}T_{rk}$.)

3. A rod of undeformed length L rotates about a point with a constant angular velocity ω as shown in Fig. 1. The reference configuration is taken to be the state when the rod is stationary, and the rod is assumed to be stress- and strain-free in this configuration. The distance of any point from the axis of rotation with respect to an observer rotating with the rod is given by x (see figure). Approximating the rod as a one-dimensional continuum, (i.e., only $\tau_{xx} \equiv \tau$, $E_{xx} \equiv E$, $F_{xx} \equiv F$ etc. are assumed to be nonzero), and taking the body forces with respect to a stationary observer to be zero, write the governing linear momentum equation for the Cauchy stress, τ, under steady-state conditions. Multiplying this relation by $J = \det\boldsymbol{F} = F$ and using the relation $\boldsymbol{\nabla}_X\cdot(\boldsymbol{F}\boldsymbol{S}) = J\boldsymbol{\nabla}_x\cdot\boldsymbol{\tau}$, transform this equation to the reference configuration. Next, assuming the constitutive relation to be given by $S = \lambda E$, where λ is a given constant and E is the Lagrangian strain, and letting $x = h(X)$, find the governing differential equation for $h(X)$ (do (40)

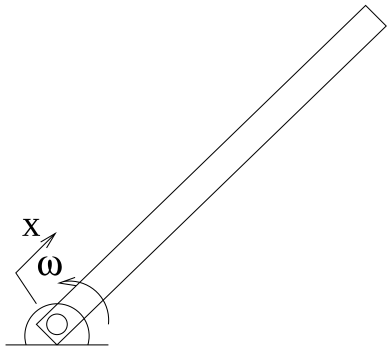

Figure 1:

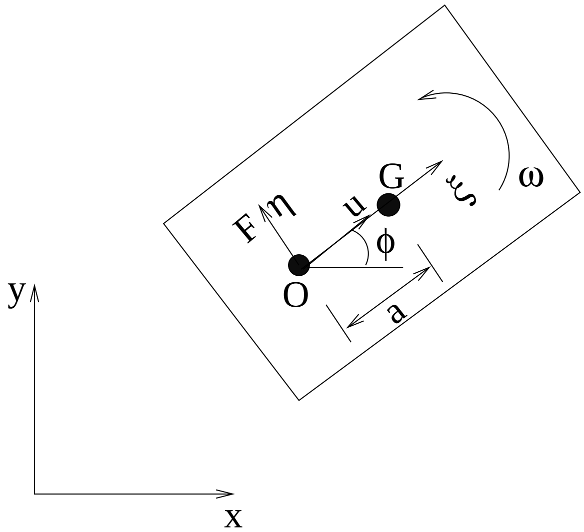

Figure 2:

not attempt to solve), and the associated boundary conditions. [Note: (i) Write only one-dimensional equations; do not write the full 3×3 matrices. (ii) The differential equation for $h(X)$ should obviously involve only known quantities defined on the reference configuration].

4. The motion of a sleigh is often modeled as shown in Fig. 2. The point O is constrained such that it cannot move along the η direction, i.e., if v denotes the velocity component along the η direction, then $v = \dot{v} = \ddot{v} = 0$. This constraint gives rise to a force F as shown. However, the sleigh is free to move in the ξ direction, and has a velocity component u in that direction (in general, $\dot{u} \neq 0$). The sleigh is also free to rotate about the point O–its angular velocity and angular accleration are given by $\omega = \dot{\phi}$, and $\dot{\omega} = \ddot{\phi}$, respectively. The center of mass, G, is at a distance of a from O along the ξ axis. Assuming the mass of the sleigh to be M, the moment of inertia about an axis perpendicular to the plane of the sleigh and passing through G, to be I, and the body forces with respect to the stationary frame xy to be (50)

negligible, find the equations of motion with respect to the xy frame. Using the mechanical energy balance

$$\frac{dK}{dt} = \hat{\boldsymbol{f}} \cdot \dot{\bar{\boldsymbol{x}}} + \boldsymbol{m}_{\bar{\boldsymbol{x}}} \cdot \boldsymbol{w}.$$

show that the kinetic energy, K, remains a constant.

5. Show that the inequality (30)

$$\lambda(\operatorname{tr} \boldsymbol{D})^2 + 2\mu \boldsymbol{D} : \boldsymbol{D} \geq 0 \quad \forall \boldsymbol{D} \in \mathrm{Sym}$$

is equivalent to the inequalities

$$\mu \geq 0, \quad 3\lambda + 2\mu \geq 0.$$

(Hint: Write $\boldsymbol{D}$ in terms of its deviatoric part $\boldsymbol{D}_0$, i.e., $\boldsymbol{D} = \boldsymbol{D}_0 + \alpha \boldsymbol{I}$, where $\alpha = \frac{1}{3}\operatorname{tr} \boldsymbol{D}$.)

Some relevant formulae

$$\begin{aligned}
\cos(\alpha + \beta) &= \cos\alpha\cos\beta - \sin\alpha\sin\beta, \\
\sin(\alpha + \beta) &= \sin\alpha\cos\beta + \sin\beta\cos\alpha.
\end{aligned}$$

Relation between the body forces, $\boldsymbol{b}$ and $\boldsymbol{b}^*$, when axial vector of $\boldsymbol{W}$ is fixed:

$$\boldsymbol{b} = \boldsymbol{Q}^t\left[\boldsymbol{b}^* - \ddot{\boldsymbol{c}}\right] - (\dot{\boldsymbol{W}} + \boldsymbol{W}^2)\boldsymbol{x} - 2\boldsymbol{W}\boldsymbol{v}.$$

Euler's equations of motion:

$$\begin{aligned}
m_1 &= J_1 \dot{w}_1^*(t) + (J_3 - J_2) w_2^* w_3^* \\
m_2 &= J_2 \dot{w}_2^*(t) + (J_1 - J_3) w_1^* w_3^* \\
m_3 &= J_3 \dot{w}_3^*(t) + (J_2 - J_1) w_1^* w_2^*.
\end{aligned}$$