Indian Institute of Science, Bangalore

ME 243: Endsemester Exam

Date: 29/1/21.
Duration: 9.15 a.m.–1.15 p.m.
Maximum Marks: 100

Instructions:
You may directly use the formulae at the back.

1. We have seen that if $\boldsymbol{Q} \in \text{Orth}^+$, then $\boldsymbol{Q}(\boldsymbol{u}\times\boldsymbol{v}) = (\boldsymbol{Q}\boldsymbol{u})\times(\boldsymbol{Q}\boldsymbol{v})$. Conversely, if $\boldsymbol{T}(\boldsymbol{u}\times\boldsymbol{v}) = (\boldsymbol{T}\boldsymbol{u})\times(\boldsymbol{T}\boldsymbol{v})$ $\forall \boldsymbol{u},\boldsymbol{v} \in \Re^3$, then is it necessarily true that $\boldsymbol{T} \in \text{Orth}^+$? Consider *all* possibilities when you try to present your solution for $\boldsymbol{T}$. (20)

2. In this problem, assume all field variables to be independent of Z. A cylinder of inner radius a, outer radius b, and length L along the z-direction is fixed rigidly at the inner boundary $r = a$, and constrained by two rigid frictionless surfaces on the top and bottom surfaces $z = 0$ and $z = L$. This cylinder is subjected to a moment M on the outer boundary $r = b$ by the application of suitable tangential tractions which are independent of θ and Z (see Fig. 1). Assume $r = \alpha R$, $z = Z$, and assume a suitable mapping for θ in terms of an unknown function. The material is incompressible, and the constitutive equation is given by $\boldsymbol{\tau} = -p\boldsymbol{I} + \mu\boldsymbol{B}$. (25)

   (a) Find the constant α and the governing differential equation for your unknown function and any other unknown field. Do not attempt to solve any of these differential equations.

   (b) State the boundary conditions for the unknown functions and the other unknown field.

3. The rigid rod shown in Fig. 2 will slide under the action of gravity assuming that the floor and the wall are frictionless. A continuum enthusiast who wants the rod to remain stationary even under the action of gravity decides to accelerate the whole setup by a constant acceleration a along the x-direction as shown in Fig. 2. (30)

   (a) Determine if the rod can remain in static equilibrium *in the accelerating frame x-y* (i.e., $\theta(t) = \theta_0$ where θ_0 is a constant), and if it can, is there a specific value of a for which it will remain in equilibrium? If there is, determine this value. For the purpose of carrying out the integrations, assume the rod to be one-dimensional, express the coordinates of a typical point (x, y) on the rod in terms the coordinate ξ which runs along the length as shown in the figure as $x = \xi\cos\theta$, $y = (L-\xi)\sin\theta$, and assume $\rho\,dV = m\,d\xi$, where m, the mass per unit length, is a constant. *Do not* use the Euler equations. Solve using first principles.

(b) For this part, assume that $a \neq 0$ is such that the rod is *not* in equilibrium in the accelerating frame of reference (i.e., $\theta(t)$ varies with time) Starting from the mechanical energy balance

$$\frac{d}{dt}\int_{V(t)} \frac{\rho \boldsymbol{v}\cdot\boldsymbol{v}}{2}\,dV = \int_{S(t)} \boldsymbol{t}\cdot\boldsymbol{v}\,dS + \int_{V(t)} \left[-\boldsymbol{\tau} : \boldsymbol{D} + \rho \boldsymbol{b}\cdot\boldsymbol{v}\right] dV,$$

derive a conservation law of the form

$$\int_{V(t)} (.)\rho\,dV = \text{constant}. \tag{1}$$

Now use Eqn. (1) and

$$\int_{V(t)} \frac{\rho \boldsymbol{v}\cdot\boldsymbol{v}}{2}\,dV = \frac{mL^3\dot{\theta}^2}{6},$$

to derive the governing equation for $\dot{\theta}$, assuming that the rod is at rest at the initial position $\theta = \theta_0$ in the accelerating frame of reference. State the initial condition for solving this equation. *Do not* attempt to solve this governing differential equation.

4. With ψ denoting the free energy, and $\boldsymbol{T}$ denoting the first Piola-Kirchhoff stress tensor, the constitutive relations for a homogeneous solid are given by (25)

$$\psi = \hat{\psi}(\boldsymbol{F}, \dot{\boldsymbol{F}}),$$
$$\boldsymbol{T} = \hat{\boldsymbol{T}}(\boldsymbol{F}, \dot{\boldsymbol{F}}),$$

where $\boldsymbol{T}$ is a continuous function of $\dot{\boldsymbol{F}}$, and where a superposed dot indicates a material derivative.

(a) Using the isothermal version of the Clausius-Duhem inequality given by

$$\left(\frac{\partial \hat{\psi}}{\partial t}\right)_X - \frac{1}{\rho_0}\boldsymbol{T} : \dot{\boldsymbol{F}} \leq 0,$$

find the equalities and inequalities (necessary and sufficient conditions) that result. In the proof for this part, *do not* attempt to construct an admissible thermodynamic process; assume that such a process can be constructed for arbitrary values of your kinematical variables. *Justify* all steps.

(b) Apply the principle of material frame-indifference (MFI) to the reduced form of $\hat{\psi}$ that you have derived in part (a). Next, using the chain rule, derive an expression for the part of the first Piola-Kirchhoff stress that depends on $\hat{\psi}$, so that MFI is automatically satisfied. State the condition for MFI for the remaining part of $\boldsymbol{T}$, but do not try to derive a reduced form in this case. You may directly use $\boldsymbol{F}^* = \boldsymbol{Q}\boldsymbol{F}$. Derive the required formulae for other kinematical variables.

Some relevant formulae

$$\boldsymbol{b} = \boldsymbol{Q}^T \left[\boldsymbol{b}^* - \ddot{\boldsymbol{c}}\right] - \dot{\boldsymbol{\Omega}} \times \boldsymbol{x} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}) - 2\boldsymbol{\Omega} \times \boldsymbol{v}.$$

$$Q_{ij} = \boldsymbol{e}_i^* \cdot \boldsymbol{e}_j,$$

$$\boldsymbol{\Omega} = \begin{bmatrix} \dot{\boldsymbol{e}}_2 \cdot \boldsymbol{e}_3 \\ \dot{\boldsymbol{e}}_3 \cdot \boldsymbol{e}_1 \\ \dot{\boldsymbol{e}}_1 \cdot \boldsymbol{e}_2 \end{bmatrix},$$

$$\boldsymbol{R} : (\boldsymbol{S}\boldsymbol{T}) = (\boldsymbol{S}^T\boldsymbol{R}) : \boldsymbol{T} = (\boldsymbol{R}\boldsymbol{T}^T) : \boldsymbol{S} = (\boldsymbol{T}\boldsymbol{R}^T) : \boldsymbol{S}^T,$$

$$F_{iJ} = \frac{h_i}{h_J}\frac{\partial \hat{\chi}_i}{\partial \eta_J}, \quad \text{no sum on } i,\ J, \quad h_i \equiv (1, r, 1), \quad h_J \equiv (1, R, 1).$$

If $\boldsymbol{\tau}$ is symmetric tensor-valued field, then the components of $\boldsymbol{\nabla}_x \cdot \boldsymbol{\tau}$ with respect to a cylindrical coordinate system are

$$(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_r = \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r},$$

$$(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_\theta = \frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r}.$$

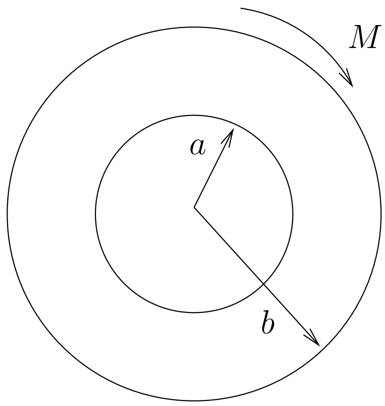

Figure 1: Hollow cylinder fixed rigidly at the inner boundary $r = a$, and subjected to a moment M on the outer boundary $r = b$.

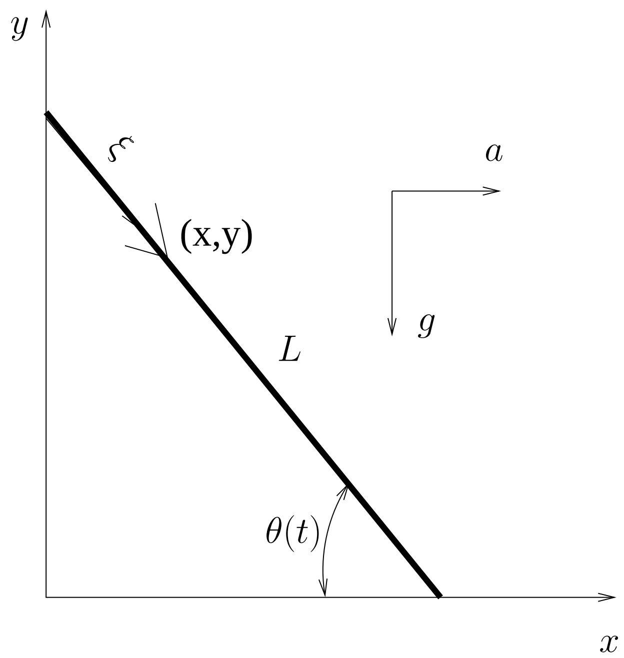

Figure 2: Problem 3.