Indian Institute of Science, Bangalore

ME 243: Endsemester Exam

Date: 10/12/21.
Duration: 9.15 a.m.–1.15 p.m.
Maximum Marks: 100

Instructions:

You may directly use the formulae at the back.

1. Let $\boldsymbol{R} \neq \boldsymbol{I}$ be a rotation with (unique) axis $\boldsymbol{e}$, and let $\boldsymbol{W} \in \text{Skw}$ with axial vector $\boldsymbol{w}$. Then show that $\boldsymbol{RW} = \boldsymbol{WR}$ if and only if (25)

$$\boldsymbol{e} = \frac{\boldsymbol{w}}{|\boldsymbol{w}|}.$$

(Hint: Operate on an arbitrary vector $\boldsymbol{u}$).

2. A hollow cylinder of length L and inner and outer radii a and b, is fixed at $R = a$, and subjected to a uniform time-dependent shear stress $s(t)\boldsymbol{e}_z$ at $R = b$ resulting in a net shear force $T(t)$, as shown in Fig. 1. The material is incompressible, and the constitutive equation is given by $\boldsymbol{\tau} = -p\boldsymbol{I} + \mu\boldsymbol{B}$. The top and bottom surfaces are traction free. (15)

   (a) Assuming $r = R$, formulate a suitable mapping for the motion in terms of an unknown function.

   (b) Find the governing partial differential equation for your unknown function and any other unknown field. Do not attempt to solve any of these differential equations.

   (c) State the boundary conditions for the unknown functions and the other unknown field. Find a relation between $s(t)$ and $T(t)$.

3. In the test, we considered the problem of a slider-crank mechanism (see Fig. 2). Imagine that there is a horizontal driving force applied to the mass at point 2 that generates the motion. At time $t = 0$, we remove the force. Let the angular velocity of the rigid circular disc, and the rigid connecting rod at the instant the force is removed be $\omega_1(0)$ and $\omega_2(0)$, respectively, and the angles that the lines 1-0 and 1-2 make with the horizontal be $\theta_1(0)$ and $\theta_2(0)$, respectively. Our goal is to find $\theta_1(t)$ and $\theta_2(t)$ after the force is removed using the given initial data mentioned above. Using the balance laws stated towards the end of the question paper, and ignoring the body forces, deduce a conservation law of the form (30)

$$\int_{V(t)} (.)\rho\, dV = \text{constant},$$

and a constraint equation, and use them to find the governing differential equations for $\theta_1(t)$ and $\theta_2(t)$. Do not attempt to solve these equations. For the purpose of carrying out the integrations, assume the rod 1-2 to be one-dimensional, express the coordinates of a typical point (x, y) on the rod in terms of the coordinate ξ which runs along the length as shown in the figure as $x = R\cos\theta_1 + \xi\cos\theta_2$ and $y = (L - \xi)\sin\theta_2$, and assume $\rho\, dV = m\, d\xi$, where m, the mass per unit length, is a constant. Similarly, assume the disc to be of unit thickness and density ρ, and the mass M of the sliding block to be lumped at point 2. You may take $\boldsymbol{\omega}_1(t) = \dot{\theta}_1(t)\boldsymbol{e}_z$ and $\boldsymbol{\omega}_2(t) = -\dot{\theta}_2(t)\boldsymbol{e}_z$.

4. The constitutive relations for a homogeneous solid are given by (30)

$$\begin{aligned}
\psi &= \tilde{\psi}(\tilde{\boldsymbol{v}}, \boldsymbol{F}, \boldsymbol{F}^T\dot{\boldsymbol{F}}, \theta_0, \boldsymbol{F}^T\boldsymbol{g}),\\
\boldsymbol{S} &= \tilde{\boldsymbol{S}}(\tilde{\boldsymbol{v}}, \boldsymbol{F}, \boldsymbol{F}^T\dot{\boldsymbol{F}}, \theta_0, \boldsymbol{F}^T\boldsymbol{g}),\\
\eta_0 &= \tilde{\eta}(\tilde{\boldsymbol{v}}, \boldsymbol{F}, \boldsymbol{F}^T\dot{\boldsymbol{F}}, \theta_0, \boldsymbol{F}^T\boldsymbol{g}),\\
\boldsymbol{q}_0 &= \tilde{\boldsymbol{q}}_0(\tilde{\boldsymbol{v}}, \boldsymbol{F}, \boldsymbol{F}^T\dot{\boldsymbol{F}}, \theta_0, \boldsymbol{F}^T\boldsymbol{g}),
\end{aligned}$$

where a superposed dot denotes a material derivative, $\tilde{\boldsymbol{v}}$ is the Lagrangian velocity vector, and $\boldsymbol{g} = \boldsymbol{\nabla}_x\theta$. Assume that each constitutive relation is a continuous function of all the kinematical variables.

(a) Using the principle of material frame-indifference (MFI) and the given constitutive relations, *derive* the list of kinematical variables on which ψ, $\boldsymbol{S}$, η_0 and $\boldsymbol{q}_0$ should depend, so that MFI is automatically satisfied. You may directly use $\boldsymbol{F}^* = \boldsymbol{Q}\boldsymbol{F}$ and $\boldsymbol{\chi}^*(X, t) = \boldsymbol{Q}(t)\boldsymbol{\chi}(\boldsymbol{X}, t) + \boldsymbol{c}(t)$. Derive the required formulae for other kinematical variables.

(b) Using the Clausius-Duhem inequality given by

$$\left(\frac{\partial\psi_0}{\partial t}\right)_{\boldsymbol{X}} + \eta_0\left(\frac{\partial\theta_0}{\partial t}\right)_{\boldsymbol{X}} - \frac{1}{\rho_0}\boldsymbol{T} : \dot{\boldsymbol{F}} + \frac{\boldsymbol{q}_0\cdot\boldsymbol{g}_0}{\rho_0\theta_0} \leq 0,$$

where $\boldsymbol{T} = \boldsymbol{F}\boldsymbol{S}$ is the first Piola-Kirchhoff stress tensor, find the equalities and inequalities (necessary and sufficient conditions) that result. In the proof for this part, *do not* attempt to construct an admissible thermodynamic process; assume that such a process can be constructed for arbitrary values of your kinematical variables (Hint: Split the stress $\boldsymbol{S}$ into an equilibrium part (defined as the stress when an appropriate kinematical variable is set to zero), and a nonequilibrium part). *Justify* all steps.

Some relevant formulae

$$\begin{aligned}
\boldsymbol{R}(\boldsymbol{w}, \alpha) &= \boldsymbol{I} + \frac{1}{|\boldsymbol{w}|}\sin\alpha\,\boldsymbol{W} + \frac{1}{|\boldsymbol{w}|^2}(1 - \cos\alpha)\boldsymbol{W}^2,\\
\boldsymbol{R} : (\boldsymbol{S}\boldsymbol{T}) &= (\boldsymbol{S}^T\boldsymbol{R}) : \boldsymbol{T} = (\boldsymbol{R}\boldsymbol{T}^T) : \boldsymbol{S} = (\boldsymbol{T}\boldsymbol{R}^T) : \boldsymbol{S}^T,\\
F_{iJ} &= \frac{h_i}{h_J}\frac{\partial\hat{\chi}_i}{\partial\eta_J}, \quad \text{no sum on } i,\ J, \quad h_i \equiv (1, r, 1), \quad h_J \equiv (1, R, 1).
\end{aligned}$$

If $\boldsymbol{\tau}$ is symmetric tensor-valued field, then the components of $\boldsymbol{\nabla}_x \cdot \boldsymbol{\tau}$ with respect to a cylindrical coordinate system are

$$\begin{aligned}
(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_r &= \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r}, \\
(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_\theta &= \frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r}, \\
(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_z &= \frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{z\theta}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_{zr}}{r}.
\end{aligned}$$

The balance laws are

$$\begin{aligned}
\frac{d}{dt}\int_{V(t)} \rho \boldsymbol{v}\, dV &= \int_{S(t)} \boldsymbol{t}\, dS + \int_{V(t)} \rho \boldsymbol{b}\, dV, \\
\frac{d}{dt}\int_{V(t)} \rho \boldsymbol{x} \times \boldsymbol{v}\, dV &= \int_{S(t)} \boldsymbol{x} \times \boldsymbol{t}\, dS + \int_{V(t)} \rho \boldsymbol{x} \times \boldsymbol{b}\, dV, \\
\frac{d}{dt}\int_{V(t)} \frac{\rho \boldsymbol{v} \cdot \boldsymbol{v}}{2}\, dV &= \int_{S(t)} \boldsymbol{t} \cdot \boldsymbol{v}\, dS + \int_{V(t)} \left[-\boldsymbol{\tau} : \boldsymbol{D} + \rho \boldsymbol{b} \cdot \boldsymbol{v}\right] dV.
\end{aligned}$$

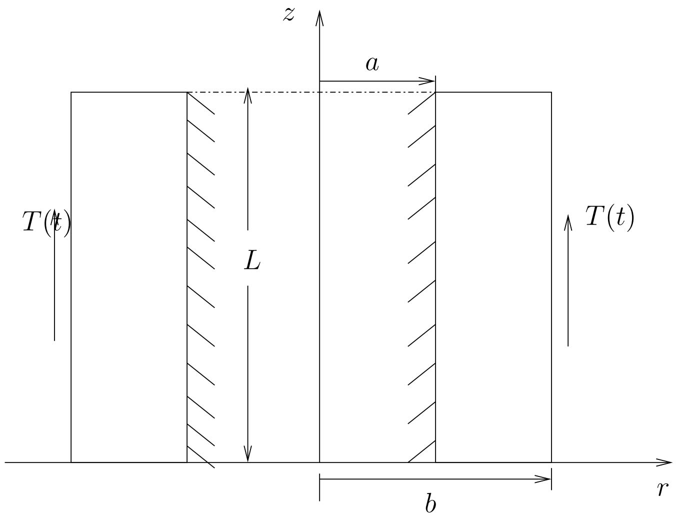

Figure 1: Hollow cylinder of length L fixed at the inner boundary $r = a$, and subjected to a shear force $T(t)$ at the outer boundary $r = b$.

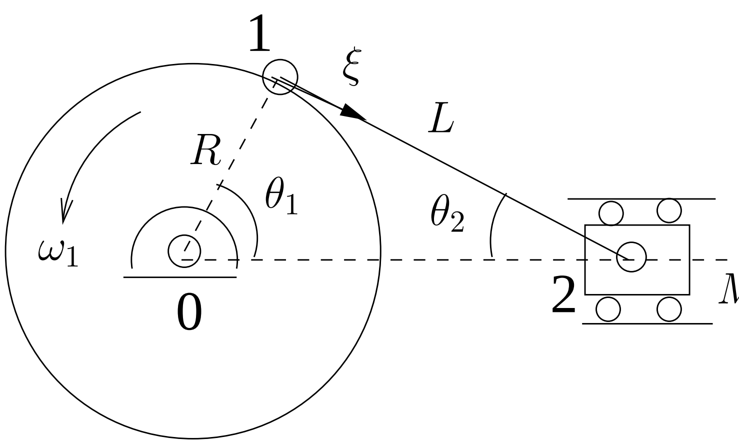

Figure 2: Slider-crank mechanism.