Indian Institute of Science, Bangalore

ME 243: Endsemester Exam

Date: 8/12/22.
Duration: 9.00 a.m.–12.00 noon
Maximum Marks: 100

Instructions:
You may directly use the formulae at the back.

1. Let $\{\boldsymbol{e}, \boldsymbol{q}, \boldsymbol{r}\}$ be an orthonormal set of vectors. (30)

   (a) Determine if the set

   $$\{\boldsymbol{e} \otimes \boldsymbol{e}, \boldsymbol{I} - \boldsymbol{e} \otimes \boldsymbol{e}, \boldsymbol{r} \otimes \boldsymbol{q} - \boldsymbol{q} \otimes \boldsymbol{r}\},$$

   is linearly independent.

   (b) A rotation can be represented as

   $$\boldsymbol{R} = \boldsymbol{e} \otimes \boldsymbol{e} + \cos\alpha(\boldsymbol{I} - \boldsymbol{e} \otimes \boldsymbol{e}) + \sin\alpha(\boldsymbol{r} \otimes \boldsymbol{q} - \boldsymbol{q} \otimes \boldsymbol{r}),$$

   Using this representation, find the dimension of $\mathrm{Lsp}\{\boldsymbol{I}, \boldsymbol{R}, \boldsymbol{R}^T\}$ for various values of α.

2. Let $\boldsymbol{R}$ be the 'rotation' part in the polar decomposition of the deformation gradient $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U}$, and let $\boldsymbol{\Omega} := \dot{\boldsymbol{R}}\boldsymbol{R}^T$, where the superposed dot denotes a material derivative. Find a relation between the vorticity tensor $\boldsymbol{W} := (\boldsymbol{L} - \boldsymbol{L}^T)/2$ and $\boldsymbol{\Omega}$ in terms of $\boldsymbol{U}$, $\dot{\boldsymbol{U}}$ etc. Derive any relations that you require on the way. State the necessary and sufficient conditions on $\dot{\boldsymbol{U}}\boldsymbol{U}^{-1}$ so that $\boldsymbol{W} = \boldsymbol{\Omega}$. Does this equality hold in case the motion is rigid? (20)

3. If μ is a constant, $\boldsymbol{L}$ is the velocity gradient, $\boldsymbol{W}$ is the vorticity tensor, $\boldsymbol{T}$ and $\boldsymbol{S}$ are the first and second Piola-Kirchhoff stress tensors, $\mathbf{C}$ is a fourth-order constant tensor, and $\boldsymbol{u}$ is the displacement field, determine if the following constitutive relations are frame-indifferent (a superposed dot represents the material derivative) (25)

   $$\begin{aligned}\dot{\boldsymbol{T}} - \boldsymbol{L}\boldsymbol{T} &= 2\mu\boldsymbol{B}\boldsymbol{F},\\ \dot{\boldsymbol{T}} - \boldsymbol{L}\boldsymbol{T} &= 2\mu\boldsymbol{D}\boldsymbol{F}^{-T},\\ \dot{\boldsymbol{T}} - \boldsymbol{T}\boldsymbol{W} &= 2\mu\boldsymbol{D}\boldsymbol{F}^{-T},\\ \dot{\boldsymbol{S}} - \mathbf{C}\dot{\boldsymbol{E}} &= \ddot{\boldsymbol{u}} \otimes \ddot{\boldsymbol{u}}.\end{aligned}$$

   You may assume $\boldsymbol{x}^* = \boldsymbol{Q}(t)\boldsymbol{x} + \boldsymbol{c}(t)$ and $\boldsymbol{F}^* = \boldsymbol{Q}(t)\boldsymbol{F}$, but derive any other relations that you need on the way.

4. A circular cylinder of initial radius a and initial length L, with frictionless contact on its top and bottom surfaces (shown as roller supports), is subjected (25)

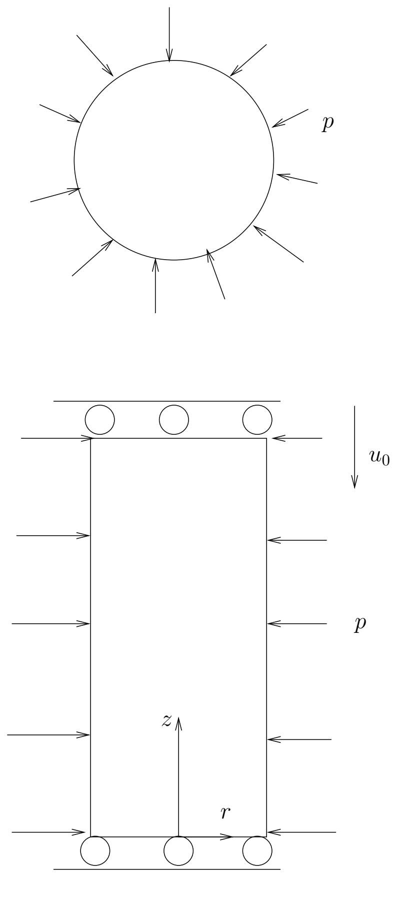

Figure 1: Cylinder subjected to uniform pressure on the lateral surface and given a prescribed displacement on its top surface.

to a uniform pressure loading on its lateral surface, while the top surface is moved by a distance u_0 as shown in Fig. 1. The constitutive equation is given by $\boldsymbol{\tau} = \mu \boldsymbol{B}$, where μ is a constant. By assuming the radial displacement field to be independent of Z and the longitudinal displacement field to be independent of R, find the governing equations for the unknown functions. State the appropriate boundary conditions (which have to be stated precisely without using any assumptions such as those made in linearized elasticity). *Do not* attempt to solve the governing equations that result for your unknown fields except for the longitudinal mapping function which you should solve for explicitly. Find the normal force exerted on the top surface in the deformed configuration explicitly.

Some relevant formulae

$$F_{iJ} = \frac{h_i}{h_J} \frac{\partial \hat{\chi}_i}{\partial \eta_J}, \quad \text{no sum on } i,\ J, \quad h_i \equiv (1, r, 1), \quad h_J \equiv (1, R, 1).$$

If $\boldsymbol{\tau}$ is symmetric tensor-valued field, then the components of $\boldsymbol{\nabla}_x \cdot \boldsymbol{\tau}$ with respect to a cylindrical coordinate system are

$$\begin{aligned}
(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_r &= \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r}, \\
(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_\theta &= \frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r}, \\
(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_z &= \frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r} \frac{\partial \tau_{z\theta}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_{zr}}{r}.
\end{aligned}$$