Indian Institute of Science, Bangalore

ME 243: Endsemester Exam

Date: 8/12/23.
Duration: 9.00 a.m.–12.00 noon
Maximum Marks: 100

Instructions:
You may directly use the formulae at the back.

1. Let $\boldsymbol{W} \in \text{Skw}$. Determine if (35)

   (a) $\{\boldsymbol{I}, \boldsymbol{W}, \boldsymbol{W}^2\}$ is a linearly independent set.

   (b) $\text{Orth}^+ \subset \text{Lsp}\{\boldsymbol{I}, \boldsymbol{W}, \boldsymbol{W}^2\}$.

   (c) $\text{Sym} \subset \text{Lsp}\{\boldsymbol{I}, \boldsymbol{W}, \boldsymbol{W}^2\}$.

   (d) $\text{Orth}^+ \cap \text{Sym} - \{\boldsymbol{I}\} \subset \text{Lsp}\{\boldsymbol{I}, \boldsymbol{W}, \boldsymbol{W}^2\}$.

2. Let $\boldsymbol{F} = \boldsymbol{\nabla}_X \boldsymbol{\chi}$ denote the deformation gradient. (25)

   (a) Using indicial notation, evaluate $\boldsymbol{\nabla} \times \boldsymbol{F}$.

   (b) By taking into account all possible constraints on the deformation gradient, determine if the following expressions are valid deformation gradients

   - $\boldsymbol{F} = \boldsymbol{c} \otimes \boldsymbol{X}$, where $\boldsymbol{c}$ is a constant vector.
   - The deformation gradient corresponding to the motion $\boldsymbol{\chi} = [\boldsymbol{I} - 2\boldsymbol{e} \otimes \boldsymbol{e}]\boldsymbol{X} + \boldsymbol{c}(t)$, where $\boldsymbol{e}$ is a unit vector.
   - $\boldsymbol{F} = \textbf{diag}[Z^2, X^2, Y^2]$.

3. Assume isothermal conditions throughout this problem, and the material to be homogeneous. Let the constitutive equation for the first Piola-Kirchhoff stress be given by (40)

$$\boldsymbol{T} = \gamma_{-1}\boldsymbol{F}^{-T} + \gamma_0 \boldsymbol{F} + \gamma_1 \boldsymbol{F}\boldsymbol{C}, \tag{1}$$

where γ_{-1}, γ_0 and γ_1 are constants. Let the material be internally constrained such that $\det \boldsymbol{V} = 1$, where $\boldsymbol{V} = \sqrt{\boldsymbol{B}}$, and assume that the free energy ψ_0 depends only on $\boldsymbol{F}$.

   (a) Starting from the isothermal version of the Clausius-Duhem inequality

$$\left(\frac{\partial \psi_0}{\partial t}\right)_{\boldsymbol{X}} - \frac{1}{\rho_0}\boldsymbol{T} : \dot{\boldsymbol{F}} \leq 0,$$

   show how the constitutive equation in Eqn. (1) would need to be modified to account for the internal constraint (you may assume that an admissible process with arbitrary $\dot{\boldsymbol{F}}$ can be constructed).

   (b) Does the modified constitutive relation satisfy material frame-indifference?

(c) Assume that $\gamma_1 = 0$ in your modified constitutive relation in the remaining part of this problem, and assume body forces to be zero. The base of a circular cylinder of radius a and length L is fixed, while its top surface is subjected to tractions that result in a torque T and a normal force F_z. The lateral surface is traction free.

i. With α denoting a constant, if $\boldsymbol{\chi}(\boldsymbol{X}, t)$ is given by

$$
\begin{aligned}
x &= X\cos(\alpha Z) - Y\sin(\alpha Z),\\
y &= X\sin(\alpha Z) + Y\cos(\alpha Z),\\
z &= dZ,
\end{aligned}
$$

then determine d such that the internal constraint is obeyed.

ii. If the deformation gradient is of the form

$$
\boldsymbol{F} = \begin{bmatrix} \cos(\alpha Z) & -\sin(\alpha Z) & ... \\ \sin(\alpha Z) & \cos(\alpha Z) & ... \\ 0 & 0 & d \end{bmatrix},
$$

then

$$
\boldsymbol{F}^{-T} = \begin{bmatrix} \cos(\alpha Z) & -\sin(\alpha Z) & 0 \\ \sin(\alpha Z) & \cos(\alpha Z) & 0 \\ \alpha Y & -\alpha X & \frac{1}{d} \end{bmatrix}.
$$

Using the above results, and the governing equation and boundary conditions, determine the unknown fields in your stress field.

iii. Find an explicit expression for the total normal force F_z on the top surface of the cylinder.

iv. Give a systematic procedure for finding the relation between the total torque T and the twist per unit length α. *Do not* attempt to carry out the detailed calculations, but all details (including limits of integration) must be precisely laid out.

Some relevant formulae

$$
\begin{aligned}
\boldsymbol{T} &= \boldsymbol{\tau}(\mathbf{cof}\,\boldsymbol{F}),\\
\boldsymbol{\nabla}_X \cdot \boldsymbol{T} &= \boldsymbol{0},\\
\boldsymbol{t}_0 &= \boldsymbol{T}\boldsymbol{n}_0,\\
\boldsymbol{n} &= \frac{(\mathbf{cof}\,\boldsymbol{F})\boldsymbol{n}_0}{|(\mathbf{cof}\,\boldsymbol{F})\boldsymbol{n}_0|},\\
dS &= |(\mathbf{cof}\,\boldsymbol{F})\boldsymbol{n}_0|\,dS_0,\\
\boldsymbol{t}_0(\boldsymbol{X}, t, \boldsymbol{n}_0) &= |(\mathbf{cof}\,\boldsymbol{F})\boldsymbol{n}_0|\,\boldsymbol{t}(\boldsymbol{\chi}(\boldsymbol{X}, t), t, \boldsymbol{n}),\\
\boldsymbol{R}(\boldsymbol{w}, \alpha) &= \boldsymbol{I} + \frac{1}{|\boldsymbol{w}|}\sin\alpha\,\boldsymbol{W} + \frac{1}{|\boldsymbol{w}|^2}(1-\cos\alpha)\boldsymbol{W}^2,\\
\boldsymbol{W} &= |\boldsymbol{w}|\,(\boldsymbol{r}\otimes\boldsymbol{q} - \boldsymbol{q}\otimes\boldsymbol{r}),\\
(\boldsymbol{\nabla}\times\boldsymbol{T})_{ij} &= \epsilon_{irs}\frac{\partial T_{js}}{\partial x_r}.
\end{aligned}
$$