Indian Institute of Science, Bangalore

ME 243: Endsemester Exam

Date: 28/11/24.
Duration: 9.00 a.m.–12.00 noon
Maximum Marks: 100

Instructions:

You may directly use the formulae at the back.

1. If $\boldsymbol{T} : (\boldsymbol{I} + \boldsymbol{u} \otimes \boldsymbol{v} - \boldsymbol{v} \otimes \boldsymbol{u}) = 0$, where $\boldsymbol{u}$ and $\boldsymbol{v}$ are arbitrary vectors, find the most general form of $\boldsymbol{T}$. Your answer should be in the form of a single expression. For example, if your answer is that $\boldsymbol{T}$ should be such that $\det \boldsymbol{T} = 1$, you should write it as $(\det \boldsymbol{T})^{-1/3}\boldsymbol{T}$, so that any 'constraints' are automatically satisfied by your solution. (25)

2. An elastic rod is constrained within a rigid sleeve which oscillates with a *given* $\theta(t)$ in the $\boldsymbol{e}_1^*$-$\boldsymbol{e}_2^*$ plane, and hence with a given angular velocity $\dot{\theta}\boldsymbol{e}_3$ (see Fig. 1) The elastic rod can slide frictionlessly within the sleeve along the $\boldsymbol{e}_1$ direction. The (undeformed) length of the rod is L before the start of the motion. The gravitational body force g acts along the $\boldsymbol{e}_1^*$ direction. We are interested in finding the governing equation of motion along the $\boldsymbol{e}_1$ direction along with the appropriate boundary conditions (you do not need to state the initial conditions). Assume $x_2 = X_2$, $x_3 = X_3$ and $x_1 = \chi(X, t)$, so that this problem is effectively a '1-D' problem along the $\boldsymbol{e}_1$ direction. Let $S := S_{11}(E) = \lambda E_{11}$ be the constitutive relation, where λ is a constant. Find the governing equation for $\chi(X, t)$ along with the boundary conditions at $X = 0$ and at $X = L$. *Do not* attempt to solve this equation. (25)

3. A rigid cylindrical container of radius a containing an incompressible Newtonian fluid is rotated about the $\boldsymbol{e}_z$ axis with constant angular velocity $\omega\boldsymbol{e}_z$ as shown in Fig. 2. Let the constitutive equation be given by $\boldsymbol{\tau} = -p(\boldsymbol{x}, t)\boldsymbol{I} + 2\mu\boldsymbol{D}$. Let the coordinate frame be the *fixed* r-θ-z frame shown in the figure; thus, the only body force in this stationary frame of reference is $-g\boldsymbol{e}_z$ (you should solve this problem with respect to this fixed frame of reference). The velocity field that automatically satisfies the continuity equation $\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0$ is given by $v_r = 0$, (20)

$$v_\theta = c_1 r + \frac{c_2}{r},$$

$$v_z = c_3,$$

where c_1, c_2 and c_3 are constants.

   (a) Using the appropriate boundary and symmetry conditions, determine the constants in the velocity field

   (b) Assuming that the top surface of the fluid is traction free, find the equation of the free surface.

4. Let the free energy, second-Piola Kirchhoff stress, entropy and heat flux vectors be given by (30)

$$\begin{aligned}\rho_0\hat{\psi} &= c_1(\operatorname{tr}\boldsymbol{E})^2 + c_2\operatorname{tr}(\boldsymbol{E}^2) - c_3\alpha(\operatorname{tr}\boldsymbol{E})(\theta_0-\theta_R) + c_4\boldsymbol{g}_0\cdot\boldsymbol{g}_0 \\ &\quad + \rho_0 c\left(\theta_0-\theta_R-\theta_0\log\frac{\theta_0}{\theta_R}\right), \\ \boldsymbol{S} &= \boldsymbol{S}_0(\boldsymbol{E},\theta_0) + c_5\dot{\boldsymbol{E}}, \\ \eta_0 &= \hat{\eta}(\boldsymbol{E},\theta_0,\boldsymbol{g}_0), \\ \boldsymbol{q}_0 &= (c_6\boldsymbol{C}^{-1} + c_7\boldsymbol{E})\boldsymbol{g}_0,\end{aligned}$$

where θ_0 denotes the actual temperature, θ_R is the reference temperature, $g_0 = \boldsymbol{\nabla}_X\theta_0$ and c_1-c_7 are constants. Starting from

$$\left(\frac{\partial\psi_0}{\partial t}\right)_{\boldsymbol{X}} + \eta_0\left(\frac{\partial\theta_0}{\partial t}\right)_{\boldsymbol{X}} - \frac{1}{\rho_0}\boldsymbol{S}:\dot{\boldsymbol{E}} + \frac{\boldsymbol{q}_0\cdot\boldsymbol{g}_0}{\rho_0\theta_0} \le 0,$$

and assuming that a suitable thermodynamically admissible process can be constructed, find

(a) Expressions for $S_0(\boldsymbol{E},\theta_0)$ and η.

(b) Using the fact that $S_0(\boldsymbol{E},\theta_0)$ should agree with the constitutive relation for a St Venant-Kirchhoff material $\boldsymbol{S} = \lambda(\operatorname{tr}\boldsymbol{E})\boldsymbol{I} + 2\mu\boldsymbol{E}$ when $\theta_0 = \theta_R$, and that $\boldsymbol{S} = \boldsymbol{0}$ when $\boldsymbol{E} = \alpha(\theta_0-\theta_R)\boldsymbol{I}$ and $\dot{\boldsymbol{E}} = \boldsymbol{0}$, find either the values of the constants c_1-c_7, or restrictions to be imposed on them (e.g., some constant has to be greater than, or less than, or equal to zero). For those constants that are expressed in terms of (λ,μ), no restrictions need be found.

(c) Determine whether the final constitutive relations for $\boldsymbol{S}$, η and $\boldsymbol{q}_0$ that you have obtained are frame-indifferent.

Some relevant formulae

$$\boldsymbol{\nabla}_X\cdot(\boldsymbol{F}\boldsymbol{S}) + \rho_0\boldsymbol{b}_0 = \rho_0\frac{\partial^2\boldsymbol{\chi}}{\partial t^2},$$

$$\boldsymbol{t}_0 = \boldsymbol{T}\boldsymbol{n}_0,$$

$$\boldsymbol{n} = \frac{(\mathbf{cof}\,\boldsymbol{F})\boldsymbol{n}_0}{|(\mathbf{cof}\,\boldsymbol{F})\boldsymbol{n}_0|},$$

$$dS = |(\mathbf{cof}\,\boldsymbol{F})\boldsymbol{n}_0|\,dS_0,$$

$$\boldsymbol{t}_0(\boldsymbol{X},t,\boldsymbol{n}_0) = |(\mathbf{cof}\,\boldsymbol{F})\boldsymbol{n}_0|\,\boldsymbol{t}(\boldsymbol{\chi}(\boldsymbol{X},t),t,\boldsymbol{n}),$$

$$\boldsymbol{b}_0(\boldsymbol{X},t) = \boldsymbol{b}(\boldsymbol{\chi}(\boldsymbol{X},t),t),$$

$$\boldsymbol{b} = \boldsymbol{Q}^T(\boldsymbol{b}^* - \ddot{\boldsymbol{c}}) - \dot{\boldsymbol{\Omega}}\times\boldsymbol{x} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{x}) - 2\boldsymbol{\Omega}\times\boldsymbol{v}.$$

$$Q_{ij} = \boldsymbol{e}_i^*\cdot\boldsymbol{e}_j,$$

$$\boldsymbol{\Omega} = \begin{bmatrix}\dot{\boldsymbol{e}}_2\cdot\boldsymbol{e}_3\\ \dot{\boldsymbol{e}}_3\cdot\boldsymbol{e}_1\\ \dot{\boldsymbol{e}}_1\cdot\boldsymbol{e}_2\end{bmatrix},$$

$$\boldsymbol{W} = |\boldsymbol{w}|\,(\boldsymbol{r}\otimes\boldsymbol{q} - \boldsymbol{q}\otimes\boldsymbol{r}),$$

$$D_{rr} = \frac{\partial v_r}{\partial r}, \qquad D_{r\theta} = \frac{1}{2}\left[\frac{1}{r}\frac{\partial v_r}{\partial \theta} + r\frac{\partial}{\partial r}\left(\frac{v_\theta}{r}\right)\right],$$

$$D_{\theta\theta} = \frac{1}{r}\left(\frac{\partial v_\theta}{\partial \theta} + v_r\right), \qquad D_{\theta z} = \frac{1}{2}\left(\frac{\partial v_\theta}{\partial z} + \frac{1}{r}\frac{\partial v_z}{\partial \theta}\right),$$

$$D_{zz} = \frac{\partial v_z}{\partial z}, \qquad D_{rz} = \frac{1}{2}\left(\frac{\partial v_r}{\partial z} + \frac{\partial v_z}{\partial r}\right).$$

The momentum equations in r, θ and z directions:

$$\begin{aligned}
\frac{\partial v_r}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})v_r - \frac{v_\theta^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\boldsymbol{\nabla}^2 v_r - \frac{v_r}{r^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + b_r,\\
\frac{\partial v_\theta}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})v_\theta + \frac{v_r v_\theta}{r} &= -\frac{1}{\rho r}\frac{\partial p}{\partial \theta} + \nu\left[\boldsymbol{\nabla}^2 v_\theta - \frac{v_\theta}{r^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta}\right] + b_\theta,\\
\frac{\partial v_z}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})v_z &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\boldsymbol{\nabla}^2 v_z + b_z.
\end{aligned}$$

where $\nu = \mu/\rho$, and

$$\begin{aligned}
\boldsymbol{v}\cdot\boldsymbol{\nabla} &\equiv v_r\frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + v_z\frac{\partial}{\partial z}\\
\boldsymbol{\nabla}^2 &\equiv \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}.
\end{aligned}$$

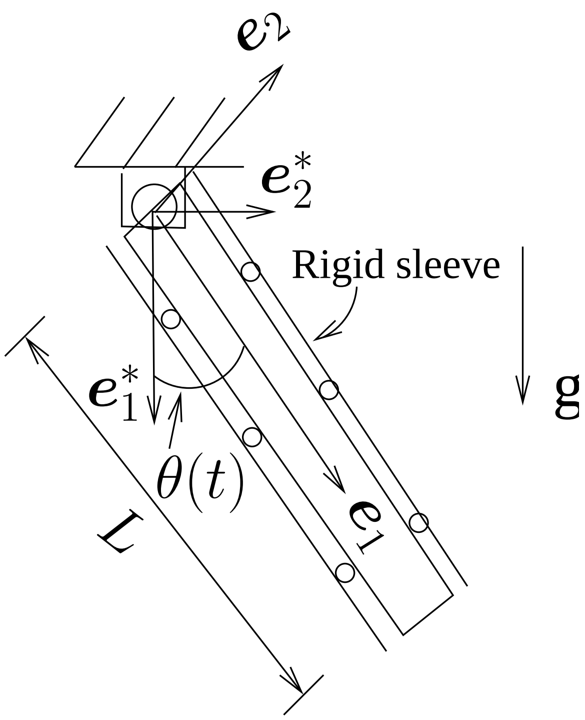

Figure 1: An elastic rod constrained in a rigid oscillating sleeve.

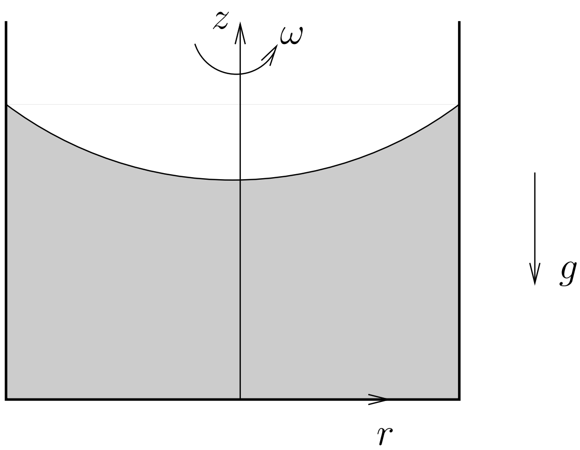

Figure 2: Fluid-filled cylinder rotating at constant speed.