Indian Institute of Science, Bangalore

ME 243: Endsemester Exam

Date: 1/12/25.
Duration: 9.00 a.m.–12.00 noon
Maximum Marks: 100

Instructions:

You may directly use the formulae at the back.

1. If (20)

$$[\boldsymbol{T}\boldsymbol{u}, \boldsymbol{T}\boldsymbol{v}, \boldsymbol{w}] = \left[\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{T}^T\boldsymbol{w}\right] \quad \forall \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in V,$$

and if $\boldsymbol{T} \neq \boldsymbol{0}$, then find the most general form of $\boldsymbol{T}$. Justify each step.

2. A small rigid body of mass M slides along a frictionless circular wire of radius R which rotates with angular velocity $\omega \boldsymbol{e}_y$ (which is equal to $\boldsymbol{\Omega}$) and angular acceleration $\dot{\omega}\boldsymbol{e}_y$ (which is equal to $\dot{\boldsymbol{\Omega}}$) as shown in Fig. 1. The gravitational body force with respect to a *fixed* frame of reference is $-g\boldsymbol{e}_y$. Assuming that the body force is a constant over the material volume occupied by the body (so that $\int_V \rho \boldsymbol{b}\, dV = M\boldsymbol{b}$), and starting from any of the balance laws of your choice for a material volume, *derive* the governing equation for the angular position $\theta(t)$ of the center of mass of the body. The initial conditions are given by $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \alpha$. Multiply your governing equation by $\dot{\theta}$ and integrate once, with the integration constant determined using the initial conditions. (25)

3. A circular hole of radius b in an unbounded domain is subjected to a shear traction $t_\theta = s_0$. Assume that the deformation is independent of Z. Further assume $r = \alpha R$, $z = Z$, and assume a suitable mapping for θ in terms of an unknown function. The material is incompressible, and the constitutive equation is given by $\boldsymbol{\tau} = -p\boldsymbol{I} + \mu \boldsymbol{B}$. (25)

   (a) Find the constant α, and the governing differential equation for your unknown function and any other unknown field.

   (b) State the boundary conditions for your unknown function and the other unknown field.

   (c) Solve for your unknown function and any other unknown field.

4. With ψ denoting the free energy, and $\boldsymbol{\tau}$ denoting the Cauchy stress tensor, the constitutive relations for a homogeneous, isotropic solid are given by (30)

$$\psi = \hat{\psi}(\boldsymbol{F}, \boldsymbol{L}),$$
$$\boldsymbol{\tau} = \hat{\boldsymbol{\tau}}(\boldsymbol{F}, \boldsymbol{L}),$$

where $\boldsymbol{\tau}$ is a continuous function of $\boldsymbol{L}$.

(a) If $\boldsymbol{S} \in \mathrm{Sym}$, $\boldsymbol{T} \in \mathrm{Lin}$ and $\boldsymbol{T}_s = (\boldsymbol{T} + \boldsymbol{T}^T)/2$, then show that $\boldsymbol{S} : \boldsymbol{T} = \boldsymbol{S} : \boldsymbol{T}_s$.

(b) Starting from $\dot{\boldsymbol{F}} = \boldsymbol{L}\boldsymbol{F}$, and with $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$, derive an expression for $\dot{\boldsymbol{B}}$ in terms of $\boldsymbol{F}$ and $\boldsymbol{L}$. Similarly, derive an expression for $\boldsymbol{D}$ as a function of $\dot{\boldsymbol{F}}$ and $\boldsymbol{F}$.

(c) Using $\boldsymbol{F}^* = \boldsymbol{Q}\boldsymbol{F}$, derive an expression for $\boldsymbol{L}^*$ in terms of $\boldsymbol{L}$.

(d) Successively applying material frame-indifference and the condition for isotropy, carry out appropriate reductions in the form for $\hat{\psi}$. (e.g., dependence on $\boldsymbol{F}$ reduces to a dependence on $\boldsymbol{U}$ or $\boldsymbol{V}$.). Provide detailed arguments for each such reduction. If you are using any theorem covered in class, cite the theorem (Hint: You can express functions of $\boldsymbol{U}$ and $\boldsymbol{V}$ as functions of $\boldsymbol{C}$ and $\boldsymbol{B}$, respectively, to simplify your analysis).

(e) Using the isothermal version of the Clausius-Duhem inequality given by

$$\dot{\psi} - \frac{1}{\rho}\boldsymbol{\tau} : \boldsymbol{L} \leq 0,$$

find the equalities and inequalities (necessary and sufficient conditions) that result. In the proof for this part, *do not* attempt to construct an admissible thermodynamic process; assume that such a process can be constructed for arbitrary values of your kinematical variables. *Justify* all steps.

Some relevant formulae

$$\frac{\partial I_1}{\partial \boldsymbol{T}} = \boldsymbol{I},$$

$$\frac{\partial I_2}{\partial \boldsymbol{T}} = (\mathrm{tr}\,\boldsymbol{T})\boldsymbol{I} - \boldsymbol{T}^T,$$

$$\frac{\partial I_3}{\partial \boldsymbol{T}} = \mathbf{cof}\,\boldsymbol{T},$$

$$\boldsymbol{b} = \boldsymbol{Q}^T(\boldsymbol{b}^* - \ddot{\boldsymbol{c}}) - \dot{\boldsymbol{\Omega}} \times \boldsymbol{x} - \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{x}) - 2\boldsymbol{\Omega} \times \boldsymbol{v},$$

$$Q_{ij} = \boldsymbol{e}_i^* \cdot \boldsymbol{e}_j,$$

$$\boldsymbol{R} : (\boldsymbol{S}\boldsymbol{T}) = (\boldsymbol{S}^T\boldsymbol{R}) : \boldsymbol{T} = (\boldsymbol{R}\boldsymbol{T}^T) : \boldsymbol{S} = (\boldsymbol{T}\boldsymbol{R}^T) : \boldsymbol{S}^T,$$

$$F_{iJ} = \frac{h_i}{h_J}\frac{\partial \hat{\chi}_i}{\partial \eta_J}, \quad \text{no sum on } i,\ J, \quad h_i \equiv (1, r, 1), \quad h_J \equiv (1, R, 1).$$

If $\boldsymbol{\tau}$ is symmetric tensor-valued field, then the components of $\boldsymbol{\nabla}_x \cdot \boldsymbol{\tau}$ with respect to a cylindrical coordinate system are

$$(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_r = \frac{\partial \tau_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{r\theta}}{\partial \theta} + \frac{\partial \tau_{rz}}{\partial z} + \frac{\tau_{rr} - \tau_{\theta\theta}}{r},$$

$$(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_\theta = \frac{\partial \tau_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{2\tau_{r\theta}}{r},$$

$$(\boldsymbol{\nabla} \cdot \boldsymbol{\tau})_z = \frac{\partial \tau_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \tau_{z\theta}}{\partial \theta} + \frac{\partial \tau_{zz}}{\partial z} + \frac{\tau_{zr}}{r}.$$

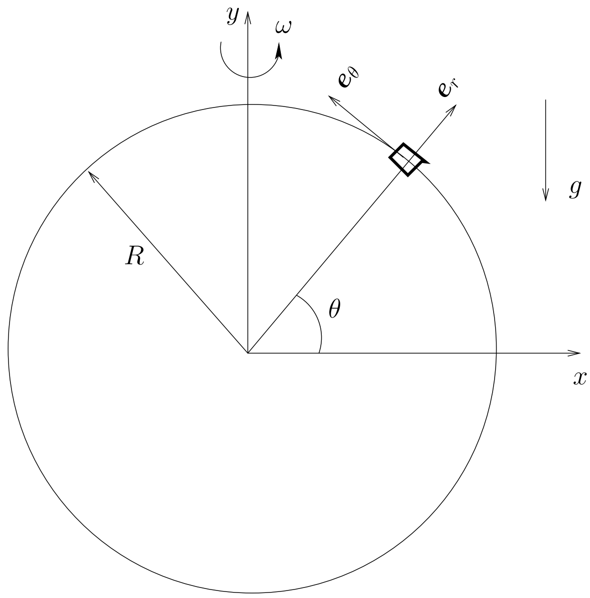

Figure 1: A small body sliding on a frictionless wire.