# Indian Institute of Science, Bangalore

# ME 243: Endsemester Exam

**Date:** 8/12/01.
**Duration:** 9.30 p.m.–12.30 p.m.
**Maximum Marks:** 200

1. Show that (20)

$$(\boldsymbol{u}, \boldsymbol{T}\boldsymbol{u}) = 0 \quad \forall \boldsymbol{u} \in V,$$

if and only if $\boldsymbol{T} \in \mathrm{Skw}$. (Hint: Split $\boldsymbol{T}$ as $\boldsymbol{T}_s + \boldsymbol{T}_{ss}$.)

2. Let $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ be skew-symmetric tensors with axial vectors $\boldsymbol{w}_1$ and $\boldsymbol{w}_2$. Show that (30)

$$\boldsymbol{W}_1\boldsymbol{W}_2 = \boldsymbol{w}_2 \otimes \boldsymbol{w}_1 - (\boldsymbol{w}_1 \cdot \boldsymbol{w}_2)\boldsymbol{I},$$
$$\mathrm{tr}\,(\boldsymbol{W}_1\boldsymbol{W}_2) = -2\boldsymbol{w}_1 \cdot \boldsymbol{w}_2.$$

By making suitable choices of $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ deduce from the second relation that

$$\boldsymbol{W} : \boldsymbol{W} = 2\boldsymbol{w} \cdot \boldsymbol{w}.$$

3. Show that $\boldsymbol{C} = \boldsymbol{F}^t\boldsymbol{F}$ and $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^t$ have the same eigenvalues. Given that the explicit formula for $\boldsymbol{R}$ in the polar decomposition $\boldsymbol{F} = \boldsymbol{R}\boldsymbol{U} = \boldsymbol{V}\boldsymbol{R}$ is (40)

$$\boldsymbol{R} = \boldsymbol{F}\left[G_1\boldsymbol{I} + G_2\boldsymbol{C} + G_3\boldsymbol{C}^2\right],$$

where $\{G_1, G_2, G_3\}$ are given functions of the eigenvalues of $\boldsymbol{U}$, find explicit expressions for $\boldsymbol{U}$, $\boldsymbol{V}$, $\boldsymbol{U}^{-1}$ and $\boldsymbol{V}^{-1}$ in terms of functions of the eigenvalues of $\boldsymbol{U}$, and powers *upto second order only* of either $\boldsymbol{B}$ or $\boldsymbol{C}$. (Hint: Show that $\boldsymbol{B} = \boldsymbol{R}\boldsymbol{C}\boldsymbol{R}^t$, and use it to find $\boldsymbol{V}^{-1}$).

4. A rubber cylinder with radius $R$ and length $L$ in its natural state is rotated about its axis of symmety with constant angular speed $\omega$. The motion with respect to a coordinate system $xyz$ *fixed* to the cylinder is given by (55)

$$x_1 = \frac{X_1}{\sqrt{\lambda}},$$
$$x_2 = \frac{X_2}{\sqrt{\lambda}},$$
$$x_3 = \lambda X_3,$$

where $\lambda$ is a positive constant (to be determined). Is the motion isochoric? The rubber is incompressible and may be regarded as a *Mooney* material characterized by the constitutive relation

$$\boldsymbol{\tau} = -p\boldsymbol{I} + (\alpha + \beta\mathrm{tr}\,\boldsymbol{B})\boldsymbol{B} - \beta\boldsymbol{B}^2,$$

where $\alpha$ and $\beta$ are positive constants. Assuming that the curved boundary of the cylinder is traction-free and by finding the body forces that act in $xyz$ (assume that the body forces w.r.t. a stationary frame are zero), find the pressure $p$. Assuming further that the resultant forces on the end-faces are zero, obtain an equation from which the length of the spinning cylinder may be obtained.

5. A material has the stored energy function of the form (55)

$$\hat{W}(\boldsymbol{F}) = \frac{\alpha}{2}\boldsymbol{F} : \boldsymbol{F} + \frac{\beta}{4}\left[(\boldsymbol{F} : \boldsymbol{F})^2 - (\boldsymbol{F}\boldsymbol{F}^t) : (\boldsymbol{F}\boldsymbol{F}^t)\right] + (\det \boldsymbol{F})^{-1},$$

where $\alpha$, $\beta$ are real constants.

(a) Is $\hat{W}(\boldsymbol{F})$ frame-indifferent and isotropic? Justify.

(b) Show that the reference configuration is a natural state and that the elasticity tensor $\mathsf{C}$ defined by $\mathsf{C}[\boldsymbol{H}] = D\hat{\boldsymbol{T}}(\boldsymbol{I})[\boldsymbol{H}]$ satisfies

$$\boldsymbol{S} : \mathsf{C}[\boldsymbol{S}] > 0 \quad \forall \boldsymbol{S} \in \mathrm{Sym} - \{\boldsymbol{0}\}$$

if and only if

$$\alpha + 2\beta = 1,\ \ \alpha + \beta > 0,\ \ -\alpha + 2\beta > 0.$$

# Some relevant formulae

Relation between the body forces, $\boldsymbol{b}$ and $\boldsymbol{b}^*$, when the axial vector $\boldsymbol{w}$ of $\boldsymbol{W}$ is fixed:

$$\boldsymbol{b} = \boldsymbol{Q}^t\left[\boldsymbol{b}^* - \ddot{\mathbf{c}}\right] - \dot{\boldsymbol{w}} \times \boldsymbol{x} - \boldsymbol{w} \times \boldsymbol{w} \times \boldsymbol{x} - 2\boldsymbol{w} \times \boldsymbol{v}.$$

Expansion of $\det(\boldsymbol{R} + \boldsymbol{S})$:

$$\det(\boldsymbol{R} + \boldsymbol{S}) = \det \boldsymbol{R} + \mathbf{cof}\,\boldsymbol{R} : \boldsymbol{S} + \boldsymbol{R} : \mathbf{cof}\,\boldsymbol{S} + \det \boldsymbol{S}.$$