Indian Institute of Science, Bangalore

ME 243: Endsemester Exam

Date: 10/12/02.
Duration: 9.30 p.m.–12.30 p.m.
Maximum Marks: 200

1. Prove that (25)
$$\rho\frac{D\boldsymbol{v}}{Dt} = \frac{\partial(\rho\boldsymbol{v})}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}\otimes\boldsymbol{v}).$$
(Hint: Use both the transport theorems.)

2. Given a vector field $\boldsymbol{q} : V \to \Re^3$ over the deformed configuration V, its Piola transform is the vector field $\boldsymbol{q}^0 : V_0 \to \Re^3$ defined over the reference configuration by the relation (35)
$$\boldsymbol{q}^0(\boldsymbol{X}) = J\boldsymbol{F}^{-1}(\boldsymbol{X})\boldsymbol{q}(\boldsymbol{x}).$$
   (a) Using the relation $\boldsymbol{\nabla}_X \cdot \textbf{cof}\,\boldsymbol{F} = \boldsymbol{0}$, show that the divergences of the two vector fields are related by
$$\boldsymbol{\nabla}_X\cdot\boldsymbol{q}^0(\boldsymbol{X}) = J\boldsymbol{\nabla}_x\cdot\boldsymbol{q}(\boldsymbol{x}).$$
   (Note that this relation is similar to the relation between the divergences of tensor fields.)

   (b) Defining
$$\begin{aligned}\psi^0(\boldsymbol{X},t) &:= \psi(\boldsymbol{x},t),\\ \eta^0(\boldsymbol{X},t) &:= \eta(\boldsymbol{x},t),\\ Q_h^0(\boldsymbol{X},t) &:= Q_h(\boldsymbol{x},t),\\ \theta^0(\boldsymbol{X},t) &:= \theta(\boldsymbol{x},t),\end{aligned}$$
   and using the relations $\rho_0 = \rho J$, $\boldsymbol{T} = \boldsymbol{\tau}\,\textbf{cof}\,\boldsymbol{F}$ and $D\boldsymbol{F}/Dt = \boldsymbol{L}\boldsymbol{F}$, find the material form of the equation
$$\frac{D\eta}{Dt} = \frac{1}{\theta}\left[Q_h - \frac{1}{\rho}\boldsymbol{\nabla}\cdot\boldsymbol{q} + \frac{1}{\rho}\boldsymbol{\tau} : \boldsymbol{L} - \eta\frac{D\theta}{Dt} - \frac{D\psi}{Dt}\right].$$

3. Using the relation $\boldsymbol{x}^* = \boldsymbol{Q}(t)\boldsymbol{x} + \boldsymbol{c}(t)$, find the relation between $\boldsymbol{v}^*$ and $\boldsymbol{v}$. Using this find the relation between $\boldsymbol{L}^*$ and $\boldsymbol{L}$, and evaluate if the expression (40)
$$\frac{D\boldsymbol{\tau}}{Dt} - \boldsymbol{L}\boldsymbol{\tau} - \boldsymbol{\tau}\boldsymbol{L}^t + (\mathrm{tr}\,\boldsymbol{L})\boldsymbol{\tau}$$
is objective.

4. Let V_0 denote the reference configuration. Assuming that $\mathbf{C}$ is symmetric, and using the linearized elasticity relations (60)

$$\rho_0 \frac{\partial^2 \boldsymbol{u}}{\partial t^2} = \boldsymbol{\nabla}_X \cdot \boldsymbol{T} + \rho_0 \boldsymbol{b}^0(\boldsymbol{X}, t),$$
$$\boldsymbol{T} = \mathbf{C}[\boldsymbol{\epsilon}],$$

show that

$$\frac{d}{dt}(K+U) = \int_{V_0} \rho_0 \frac{\partial \boldsymbol{u}}{\partial t} \cdot \boldsymbol{b}(\boldsymbol{X}, t)\, dV + \int_{(S_0)_t} \frac{\partial \boldsymbol{u}}{\partial t} \cdot \bar{\boldsymbol{t}}^0\, dS.$$

where

$$K = \int_{V_0} \frac{\rho_0}{2} \frac{\partial \boldsymbol{u}}{\partial t} \cdot \frac{\partial \boldsymbol{u}}{\partial t}\, dV,$$
$$U = \int_{V_0} \tilde{W}\, dV = \frac{1}{2} \int_{V_0} \boldsymbol{\epsilon} : \mathbf{C}[\boldsymbol{\epsilon}]\, dV,$$

$\bar{\boldsymbol{t}}^0$ is the prescribed traction on $(S_0)_t$, and $\boldsymbol{u} = \bar{\boldsymbol{u}}$ is a fixed (i.e., not varying with time) prescribed displacement field on $(S_0)_u$.

5. A cylinder of radius R and length L is fixed at one end and subjected to a torque at the other end. Assuming the 3-direction to be along the axis of the cylinder, the deformation field is given by (40)

$$x_1 = X_1 \cos(\phi X_3) - X_2 \sin(\phi X_3),$$
$$x_2 = X_1 \sin(\phi X_3) + X_2 \cos(\phi X_3),$$
$$x_3 = X_3.$$

(a) Show that the deformation is isochoric (volume preserving).

(b) Assuming the body forces to be zero, and the constitutive relation to be given by

$$\boldsymbol{\tau} = -p\boldsymbol{I} + \alpha \boldsymbol{B},$$

where $p = c + \frac{1}{2}\phi^2 \alpha (R^2 - r^2)$ (c is a constant and $r^2 = x_1^2 + x_2^2$), verify that the linear momentum equations are satisfied.

(c) Enforcing the zero traction boundary condition on the lateral surface, find the expression for c.