Indian Institute of Science, Bangalore

ME 243: Endsemester Exam

Date: 10/12/04.
Duration: 2.00 p.m.–5.00 p.m.
Maximum Marks: 100

1. (a) Show that the proper orthogonal tensor (20)

$$\boldsymbol{R}(\boldsymbol{w},\alpha) = \boldsymbol{I} + \frac{1}{|\boldsymbol{w}|}\sin\alpha\,\boldsymbol{W} + \frac{1}{|\boldsymbol{w}|^2}(1-\cos\alpha)\boldsymbol{W}^2,$$

rotates any vector in the plane perpendicular to $\boldsymbol{w}$ (where $\boldsymbol{w}$ is the axial vetor of $\boldsymbol{W} \in \text{Skw}$) by an angle α.

(b) If $\beta(\boldsymbol{a}\times\boldsymbol{b})$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are arbitrary vectors, and β is a scalar, is the axial vector of the skew tensor $\boldsymbol{b}\otimes\boldsymbol{a} - \boldsymbol{a}\otimes\boldsymbol{b}$, find β.

(c) Let $\boldsymbol{u}$, $\boldsymbol{v}$ be unit vectors such that $\boldsymbol{u}\cdot\boldsymbol{v} \neq -1$. Use the above development to find an expression for the orthogonal tensor which rotates $\boldsymbol{u}$ into $\boldsymbol{v}$ about an axis perpendicular to $\boldsymbol{u}$ and $\boldsymbol{v}$. This expression should only involve $\boldsymbol{I}$, $\boldsymbol{u}\cdot\boldsymbol{v}$, $\boldsymbol{u}\otimes\boldsymbol{u}$, $\boldsymbol{u}\otimes\boldsymbol{v}$, $\boldsymbol{v}\otimes\boldsymbol{u}$ and $\boldsymbol{v}\otimes\boldsymbol{v}$.

2. A cylinder made of an incompressible material with initial radius R_0 and length L in its natural state is rotated about its axis of symmetry with constant angular speed ω. The deformation at steady state with respect to a frame of reference fixed to the rotating cylinder is (30)

$$r = aR; \quad \theta = \Theta; \quad z = fZ,$$

where a and f are positive constants, and (r,θ,z) and (R,Θ,Z) are cylindrical coordinates in the deformed and undeformed configurations, respectively. In the deformed configuration, the lateral surface of the cylinder is traction free, while the top and bottom surfaces are constrained so that the net axial force F_z is zero. The body force under steady-state conditions is given by $\boldsymbol{b} \equiv (b_r, b_\theta, b_z) = (r\omega^2, 0, 0)$, while the constitutive relation is given by $\boldsymbol{\tau} = -p\boldsymbol{I} + \beta_{-1}\boldsymbol{B}^{-1} + \beta_1\boldsymbol{B}$, where β_{-1} and β_1 are functions of the first two invariants of $\boldsymbol{B}$.

(a) Using the expression,

$$F_{iJ} = \frac{h_i}{h_J}\frac{\partial\hat{\chi}_i}{\partial\eta_J}, \quad \text{no sum on } i,\ J. \tag{1}$$

with $h_i \equiv (1,r,1)$, $h_J \equiv (1,R,1)$, and $\eta_J \equiv (R,\Theta,Z)$, and the fact that the deformation is isochoric, find $\boldsymbol{F}$ as a function of the constant f alone.

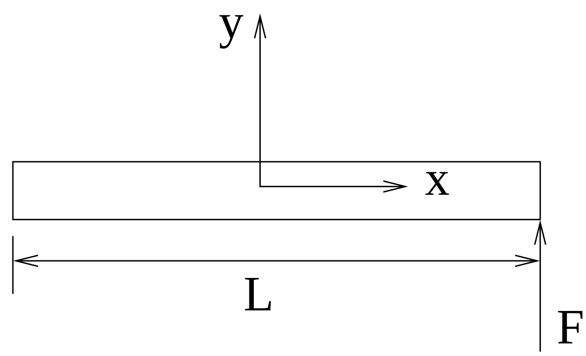

Figure 1: Bar subjected to point load F and total frictional force $\mu M g$.

(b) Using the traction boundary conditions and other relevant equations, find an expression for the pressure field, and then find expressions for the stress components.

(c) Using the fact that, in the deformed configuration, the net axial force on the top surface is zero, find an equation for determining f (*Do not* attempt to solve this equation).

(d) If $\beta_{-1} < 0$ and $\beta_1 > 0$, then deduce from this equation (without solving it) whether the cylinder's length increases, decreases or remains the same.

3. Starting from the equation of rigid motion, $\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t) = \boldsymbol{Q}(t)\boldsymbol{X} + \boldsymbol{c}(t)$, derive the formulae for the velocity and acceleration, $\boldsymbol{v}(\boldsymbol{x}, t)$ and $\boldsymbol{a}(\boldsymbol{x}, t)$, at a point with position vector $\boldsymbol{x}$ in a body undergoing rigid motion, in terms of the axial vector $\boldsymbol{w}(t)$ of $\dot{\boldsymbol{Q}}\boldsymbol{Q}^t$. Use the formula for $\boldsymbol{a}(\boldsymbol{x}, t)$ to find an expression for $\boldsymbol{a}(\boldsymbol{x}, 0)$ in the case of starting planar motion, i.e., the case where the motion is in the x-y plane, $\boldsymbol{w}(t) = w(t)\boldsymbol{e}_z$ with $w(0) = 0$, and $\boldsymbol{v}(\boldsymbol{x}, 0) = \boldsymbol{0}$. (30)

   Now consider the stationary rigid bar shown in Fig. 1 which, at time $t = 0$, is subjected to a point force $F\boldsymbol{e}_y$ at $x = L/2$, and a uniform friction force given by $-\mu M g \boldsymbol{e}_y$ so that $\int \boldsymbol{t}\, dS = (F - \mu M g)\boldsymbol{e}_y$, where μ is the coefficient of friction, M is the mass and g the gravitational acceleration. Approximating the bar in its initial position (as shown in the figure) as a one-dimensional body with $\boldsymbol{x} = x\boldsymbol{e}_x$ and $\rho\, dV = \frac{M}{L}\, dx$, find the acceleration at the point of application of the force F at time $t = 0$ by considering the entire bar as the material volume, and using the appropriate balance laws in their original integral forms, i.e., by using first principles. You may write all the relevant equations in the x-y plane.

4. The goal of this exercise is to derive the constitutive relation for an incompressible, Newtonain fluid in a purely mechanical setup (no thermal effects). Thus, let (20)

$$\psi = \hat{\psi}(\rho, \boldsymbol{L}),$$

$$\boldsymbol{\tau} = \hat{\boldsymbol{\tau}}(\rho, \boldsymbol{L}),$$

where $\hat{\boldsymbol{\tau}}$ is continuous with respect to $\boldsymbol{L}$, be the assumed constitutive relations. Using the second law of thermodynamics in the form (where a superposed dot indicates a material derivative)

$$\dot{\psi} - \frac{1}{\rho}\boldsymbol{\tau} : \boldsymbol{L} \le 0,$$

and considering the incompressibility constraint on $\boldsymbol{F}$, find the form of the constitutive relation for $\boldsymbol{\tau}$. You may assume that appropriate processes can be constructed so that arbitrary values of $\dot{\rho}$, $\dot{\boldsymbol{L}}$ etc. can be attained.

Next, using the relation $\boldsymbol{F}^* = \boldsymbol{Q}\boldsymbol{F}$, derive a relation for $\boldsymbol{L}^*$, and use this relation and the axiom of material frame-indifference to derive the final form of the constitutive relation for an incompressible, Newtonian fluid. (You may directly use any characterization theorems for isotropic functions).

Some relevant formulae

$$\begin{aligned}
\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) &= (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{w}, \\
(\boldsymbol{u} \times \boldsymbol{v}) \times \boldsymbol{w} &= (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{w})\boldsymbol{u}, \\
(\boldsymbol{u} \times \boldsymbol{v}) \cdot (\boldsymbol{u} \times \boldsymbol{v}) &= (\boldsymbol{u} \cdot \boldsymbol{u})(\boldsymbol{v} \cdot \boldsymbol{v}) - (\boldsymbol{u} \cdot \boldsymbol{v})^2, \\
(\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{c} \otimes \boldsymbol{d}) &= (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a} \otimes \boldsymbol{d}.
\end{aligned}$$

$$\begin{aligned}
\rho_0 &= \rho J, \\
\frac{DJ}{Dt} &= J\boldsymbol{\nabla} \cdot \boldsymbol{v}.
\end{aligned}$$

If $\boldsymbol{T} \in \mathrm{Lin}$, the components of $\boldsymbol{\nabla} \cdot \boldsymbol{T}$ in the cylindrical coordinate system are

$$\begin{aligned}
(\boldsymbol{\nabla} \cdot \boldsymbol{T})_r &= \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{T_{rr} - T_{\theta\theta}}{r}, \\
(\boldsymbol{\nabla} \cdot \boldsymbol{T})_\theta &= \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{T_{r\theta} + T_{\theta r}}{r}, \\
(\boldsymbol{\nabla} \cdot \boldsymbol{T})_z &= \frac{\partial T_{zr}}{\partial r} + \frac{1}{r}\frac{\partial T_{z\theta}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{zr}}{r}.
\end{aligned}$$