Indian Institute of Science, Bangalore

ME 243: Endsemester Exam

Date: 12/12/05.
Duration: 2.00 p.m.–5.00 p.m.
Maximum Marks: 100

1. If $\boldsymbol{w}$ is the axial vector of $\boldsymbol{W} \in \mathrm{Skw}$, find the axial vector of $\boldsymbol{Q}\boldsymbol{W}^3\boldsymbol{Q}^t$ in terms of $\boldsymbol{Q}$ and $\boldsymbol{w}$, where $\boldsymbol{Q} \in \mathrm{Orth}^+$. (25)

2. Derive the relation (15)
$$\frac{D\boldsymbol{F}}{Dt} = \boldsymbol{L}\boldsymbol{F}.$$
Next, using the relation $\boldsymbol{C} = \boldsymbol{F}^t\boldsymbol{F}$, relate $D\boldsymbol{C}/Dt$ and $\boldsymbol{D}$. Using $DJ/Dt = J\operatorname{tr}\boldsymbol{D}$, derive an expression for DJ/Dt in terms of J, $\boldsymbol{C}^{-1}$ and $D\boldsymbol{C}/Dt$.

3. This problem attempts to find an exact solution to the bending of a prismatic beam of rectangular cross-section with width b and height h into a region bounded by two concentric arcs as shown in Fig. 1. The material is assumed to be a St Venant-Kirchhoff material with $\nu = 0$. The neutral plane (i.e., along which $E_{XX} = 0$) is assumed to deform into an arc of radius R, and each plane Z = constant in the undeformed configuration is assumed to deform into a plane of constant radius $(R - f(Z))$, where $f(Z)$ is a function to be determined. The top and bottom surfaces are assumed to be traction free. Note that the planes X = constant are assumed to remain plane after deformation. Under these assumptions, the deformation is given by (35)
$$x = [R - f(Z)]\sin\left(\frac{X}{R}\right),$$
$$y = Y,$$
$$z = [R - f(Z)]\left[1 - \cos\left(\frac{X}{R}\right)\right] + f(Z).$$
Formulate the problem on the reference configuration as follows:
   (a) Find the deformation gradient $\boldsymbol{F}$ and Lagrangian strain $\boldsymbol{E}$.
   (b) Use the constitutive relation $\boldsymbol{S} = E\boldsymbol{E}$ to find $\boldsymbol{S}$.
   (c) Use the equations of equilibrium to find the governing differential equation for $f(Z)$. *Do not* attempt to solve this equation.
   (d) Find the appropriate boundary conditions for $f(Z)$.
   (e) Find the total force acting on the face $A'B'$ in the deformed configuration.

4. The stored energy function for a compressible neo-Hookean material is given by (with $J \equiv \det \boldsymbol{F}$) (25)

$$\tilde{W}(\boldsymbol{C}) = \frac{\lambda}{4}(J^2 - 1 - 2\ln J) + \frac{\mu}{2}(\operatorname{tr}\boldsymbol{C} - 3 - 2\ln J).$$

(a) Find an expression for $\partial J/\partial \boldsymbol{C}$ in terms of J and $\boldsymbol{C}^{-1}$.

(b) Using $\tilde{\boldsymbol{S}}(\boldsymbol{C}) = 2\partial\tilde{W}/\partial\boldsymbol{C}$, find $\tilde{\boldsymbol{S}}(\boldsymbol{C})$ as a function of $\det \boldsymbol{C}$ and $\boldsymbol{C}^{-1}$.

(c) Find the first order expansion $\tilde{\boldsymbol{S}}(\boldsymbol{I}) + D\tilde{\boldsymbol{S}}(\boldsymbol{I})[2\boldsymbol{E}]$, and compare with the constitutive relation for a St Venant-Kirchhoff material, given by $\check{\boldsymbol{S}}(\boldsymbol{E}) = \lambda(\operatorname{tr}\boldsymbol{E})\boldsymbol{I} + 2\mu\boldsymbol{E}$.

Some relevant formulae

$$\boldsymbol{W} = |\boldsymbol{w}|\,(\boldsymbol{r}\otimes\boldsymbol{q} - \boldsymbol{q}\otimes\boldsymbol{r}),$$
$$w_i = -\frac{1}{2}\epsilon_{ijk}W_{jk},$$
$$W_{ij} = -\epsilon_{ijk}w_k,$$
$$\boldsymbol{t}^0 = \boldsymbol{T}\boldsymbol{n}^0,$$
$$\boldsymbol{t}^0\,dS_0 = \boldsymbol{t}\,dS,$$
$$\det(\boldsymbol{R}+\boldsymbol{S}) = \det\boldsymbol{R} + \mathbf{cof}\,\boldsymbol{R} : \boldsymbol{S} + \boldsymbol{R} : \mathbf{cof}\,\boldsymbol{S} + \det\boldsymbol{S}.$$

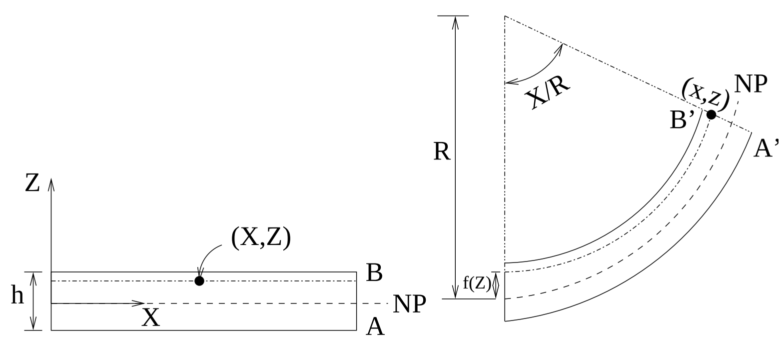

Figure 1: Bending of a prismatic beam made of a St Venant-Kirchhoff material with $\nu = 0$ into a circular arc; the neutral plane is assumed to be midway along the thickness in the undeformed configuration, i.e., at a distance $h/2$ from the top surface. The point (X, Z) deforms to (x, z) as shown.