Indian Institute of Science, Bangalore

ME 243: Endsemester Exam

Date: 12/12/06.
Duration: 2.30 p.m.–5.30 p.m.
Maximum Marks: 100

1. We saw in the test that $(\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a})/2$ is a symmetric tensor with determinant zero. Determine if every symmetric tensor with determinant zero can be represented as $(\boldsymbol{a} \otimes \boldsymbol{b} + \boldsymbol{b} \otimes \boldsymbol{a})/2$. (20)

2. If $\boldsymbol{T} \in \mathrm{Lin}$ and $\boldsymbol{u} \in V$, then show that (30)

$$(\boldsymbol{T}\boldsymbol{u}, \boldsymbol{T}\boldsymbol{u}) \leq (\boldsymbol{T}, \boldsymbol{T})(\boldsymbol{u}, \boldsymbol{u}).$$

   (Hint: Use the definition of transpose.)

3. For an elastic material, (20)

   (a) using the conditions for material frame-indifference and isotropy for the Cauchy stress tensor, *derive* the corresponding relations for the first Piola-Kirchhoff stress tensor.

   (b) using the relation $\hat{\boldsymbol{T}}(\boldsymbol{F}) = \partial \hat{W}/\partial \boldsymbol{F}$, find $\hat{\boldsymbol{T}}(\boldsymbol{F})$ corresponding to the stored-energy function

$$\hat{W}(\boldsymbol{F}) = \frac{\alpha}{4}(\boldsymbol{F} : \boldsymbol{F})^2 - (\frac{3\alpha}{4} + \frac{\beta}{2})\boldsymbol{F} : \boldsymbol{F} + \frac{\beta}{4}(\boldsymbol{F}\boldsymbol{F}^T) : (\boldsymbol{F}\boldsymbol{F}^T),$$

   where α and β are constants, and determine if the material is frame-indifferent and isotropic.

4. A circular shaft of initial radius R_0 and length L is subjected to a torque T by tractions applied to the top surface. The lateral surfaces are traction free, and the bottom surface is fixed. The Z-axis lies along the axis of the cylinder, with the origin at the center of the bottom surface. We assume a St Venant-Kirchhoff material model with $\lambda = \mu$ ($\nu = 1/4$), i.e., (30)

$$\boldsymbol{S} = \mu(\operatorname{tr} \boldsymbol{E})\boldsymbol{I} + 2\mu\boldsymbol{E}.$$

   The deformation is assumed to be of the form

$$\begin{aligned} r &= R, \\ \theta &= \Theta + dZ, \\ z &= fZ. \end{aligned}$$

Using the scale factors $h_i = (1, r, 1)$ and $h_J^0 = (1, R, 1)$, and the relation

$$F_{iJ} = \frac{h_i}{h_J^0}\frac{\partial \hat{\chi}_i}{\partial \eta_J}, \quad \text{no sum on } i,\ J,$$

where η_J denote (R, Θ, Z), find the deformation gradient. Set up equations (*but do not solve*) for determining the constants d and f, so that the above deformation is a solution to the mentioned problem. Also, determine if a normal traction has to be applied on the top surface to maintain this deformation, and if so, give an expression for it (in terms of the constants d and f).

Some relevant formulae

$$I_2(\boldsymbol{T}) = \frac{1}{2}\left[(\operatorname{tr}\boldsymbol{T})^2 - \operatorname{tr}(\boldsymbol{T}^2)\right],$$

$$\boldsymbol{\nabla}_X \cdot (\boldsymbol{F}\boldsymbol{S}) = \boldsymbol{0},$$

$$\boldsymbol{T} = \boldsymbol{\tau}\,\mathbf{cof}\,\boldsymbol{F},$$

$$\boldsymbol{t}_0 = \boldsymbol{T}\boldsymbol{n}_0 = |(\mathbf{cof}\,\boldsymbol{F})\boldsymbol{n}_0|\,\boldsymbol{t},$$

$$\mathbf{cof}\,\boldsymbol{F} = (\det \boldsymbol{F})\boldsymbol{F}^{-T} = \begin{bmatrix} F_{22}F_{33} - F_{23}F_{32} & F_{23}F_{31} - F_{21}F_{33} & F_{21}F_{32} - F_{22}F_{31} \\ F_{32}F_{13} - F_{33}F_{12} & F_{33}F_{11} - F_{31}F_{13} & F_{31}F_{12} - F_{32}F_{11} \\ F_{12}F_{23} - F_{13}F_{22} & F_{13}F_{21} - F_{11}F_{23} & F_{11}F_{22} - F_{12}F_{21} \end{bmatrix}.$$

If $\boldsymbol{T}$ is tensor-valued field, then the components of $\boldsymbol{\nabla}_x \cdot \boldsymbol{T}$ with respect to a cylindrical coordinate system are

$$(\boldsymbol{\nabla}\cdot\boldsymbol{T})_r = \frac{\partial T_{rr}}{\partial r} + \frac{1}{r}\frac{\partial T_{r\theta}}{\partial \theta} + \frac{\partial T_{rz}}{\partial z} + \frac{T_{rr} - T_{\theta\theta}}{r},$$

$$(\boldsymbol{\nabla}\cdot\boldsymbol{T})_\theta = \frac{\partial T_{\theta r}}{\partial r} + \frac{1}{r}\frac{\partial T_{\theta\theta}}{\partial \theta} + \frac{\partial T_{\theta z}}{\partial z} + \frac{T_{r\theta} + T_{\theta r}}{r},$$

$$(\boldsymbol{\nabla}\cdot\boldsymbol{T})_z = \frac{\partial T_{zr}}{\partial r} + \frac{1}{r}\frac{\partial T_{z\theta}}{\partial \theta} + \frac{\partial T_{zz}}{\partial z} + \frac{T_{zr}}{r}.$$