Indian Institute of Science, Bangalore
ME 243: Midsemester Test

Date: 25/9/08.
Duration: 3.30 p.m.–5.00 p.m.
Maximum Marks: 100

1. Let $\boldsymbol{W} \in$ Skw, and let $\boldsymbol{w}$ be its axial vector. Using the representation $\boldsymbol{W} = |\boldsymbol{w}|\,(\boldsymbol{r}\otimes\boldsymbol{q} - \boldsymbol{q}\otimes\boldsymbol{r})$, where $\boldsymbol{w}/|\boldsymbol{w}|$, $\boldsymbol{q}$ and $\boldsymbol{r}$ form an orthonormal basis, determine if $\{\boldsymbol{I}, \boldsymbol{W}, \boldsymbol{W}^2\}$ is a linearly independent set. Next determine if $\{\boldsymbol{I}, \boldsymbol{W}, \boldsymbol{W}^2, \boldsymbol{W}^3\}$ is a linearly independent set. (20)

2. Let $\boldsymbol{S} \in$ Sym, and let $\{\boldsymbol{e}_k\}$ and $\{\boldsymbol{e}_k^*\}$ denote the canonical basis of V and the principal basis of $\boldsymbol{S}$, respectively. If $\boldsymbol{Q}$ is the matrix with the eigenvectors of $\boldsymbol{S}$ along the rows, i.e., $\boldsymbol{Q} = \boldsymbol{e}_k \otimes \boldsymbol{e}_k^*$, then show that $\boldsymbol{Q} \in \text{Orth}^+$, and that $\boldsymbol{QSQ}^T$ is diagonal. (25)

   Conversely, if $\boldsymbol{Q} \in \text{Orth}^+$ diagonalizes $\boldsymbol{S}$, i.e., $\boldsymbol{QSQ}^T = \boldsymbol{\Lambda}$, where $\boldsymbol{\Lambda}$ is a diagonal matrix, then show that

   (a) $\boldsymbol{S} \in$ Sym,

   (b) $\boldsymbol{\Lambda}$ is comprised of the eigenvalues of $\boldsymbol{S}$, and

   (c) the rows of $\boldsymbol{Q}$ are the eigenvectors of $\boldsymbol{S}$.

3. Let $\boldsymbol{F}$ denote the deformation gradient. If $\boldsymbol{u}(\boldsymbol{X})$ and $\boldsymbol{v}(\boldsymbol{x})$ are related by (20)

$$\boldsymbol{u}(\boldsymbol{X}) := \boldsymbol{F}^T\boldsymbol{v}(\boldsymbol{x}),$$

   then find a relation between $\boldsymbol{F}[\boldsymbol{\nabla}_X \times \boldsymbol{u}(\boldsymbol{X})]$ and $\boldsymbol{\nabla}_x \times \boldsymbol{v}(\boldsymbol{x})$.

4. Let $\boldsymbol{C} = \boldsymbol{F}^T\boldsymbol{F} \in$ Sym be invertible, and let (35)

$$W(\boldsymbol{C}) = \frac{\kappa}{2}(J-1)^2 + \frac{\mu}{2}(\operatorname{tr}\hat{\boldsymbol{C}} - 3),$$

   where $J = \det\boldsymbol{F}$ and $\hat{\boldsymbol{C}} = (\det\boldsymbol{C})^{-1/3}\boldsymbol{C}$. Using $\boldsymbol{S}(\boldsymbol{C}) = 2\partial W/\partial\boldsymbol{C}$, find an expression for $\boldsymbol{S}(\boldsymbol{C})$. Next, find $\boldsymbol{S}(\boldsymbol{I}) + D\boldsymbol{S}(\boldsymbol{I})[2\boldsymbol{E}]$. (You may first find $\boldsymbol{S}(\boldsymbol{I}) + D\boldsymbol{S}(\boldsymbol{I})[\boldsymbol{U}]$, and then put $\boldsymbol{U} = 2\boldsymbol{E}$).

Some relevant formulae

$$(\boldsymbol{a}\otimes\boldsymbol{b})(\boldsymbol{c}\otimes\boldsymbol{d}) = (\boldsymbol{b}\cdot\boldsymbol{c})\boldsymbol{a}\otimes\boldsymbol{d},$$
$$\det(\boldsymbol{T}+\boldsymbol{U}) = \det\boldsymbol{T} + \textbf{cof}\,\boldsymbol{T} : \boldsymbol{U} + \textbf{cof}\,\boldsymbol{U} : \boldsymbol{T} + \det\boldsymbol{U},$$
$$\det\boldsymbol{T} = \frac{1}{6}\epsilon_{ijk}\epsilon_{pqr}T_{ip}T_{jq}T_{kr},$$
$$\epsilon_{pqr}(\det\boldsymbol{T}) = \epsilon_{ijk}T_{ip}T_{jq}T_{kr} = \epsilon_{ijk}T_{pi}T_{qj}T_{rk}.$$