Indian Institute of Science, Bangalore
ME 243: Midsemester Test

Date: 24/9/09.
Duration: 11.30 a.m.–1.00 p.m.
Maximum Marks: 100

1. Answer the following: (30)

   (a) If $\boldsymbol{S}\boldsymbol{T}_1 = \boldsymbol{S}\boldsymbol{T}_2\ \forall \boldsymbol{S} \in \text{Sym}$, does this imply $\boldsymbol{T}_1 = \boldsymbol{T}_2$. If not, give a counterexample.

   (b) If $\boldsymbol{Q}\boldsymbol{T}_1 = \boldsymbol{Q}\boldsymbol{T}_2\ \forall \boldsymbol{Q} \in \text{Orth}^-$, does this imply $\boldsymbol{T}_1 = \boldsymbol{T}_2$. If not, give a counterexample.

   (c) If $\boldsymbol{W}\boldsymbol{T}_1 = \boldsymbol{W}\boldsymbol{T}_2\ \forall \boldsymbol{W} \in \text{Skw}$, does this imply $\boldsymbol{T}_1 = \boldsymbol{T}_2$. If not, give a counterexample. (Hint: You can prove it in your own way, but one of the ways could be to see if equality of tensors can be invoked.)

2. A tensor $\boldsymbol{P}_i$ is a projection if it satisfies the relation $\boldsymbol{P}_i\boldsymbol{P}_j = \delta_{ij}\boldsymbol{P}_j$ (no sum on j). Let $\boldsymbol{S} = \sum_{i=1}^{2} \lambda_i \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i^*$ be the spectral resolution in dimension 2, and let the λ_i be distinct. (25)

   (a) Evaluate if $\boldsymbol{P}_i := \boldsymbol{e}_i^* \otimes \boldsymbol{e}_i^*$, $i = 1, 2$ are projections.

   (b) Find an explicit expression for $\boldsymbol{P}_i$, $i = 1, 2$ in terms of $\boldsymbol{S}$, $\boldsymbol{I}$, and the (distinct) eigenvalues λ_i of $\boldsymbol{S}$.

   (c) Let a sequence be defined by the recurrence relation $S_{n+1} = 2S_n + S_{n-1}$. Assuming $S_1 = S_2 = 1$, the first few members of this sequence are $1, 1, 3, 7, 17, 41, \ldots$. Our goal is to find an explicit formula for S_n. First write

   $$\begin{bmatrix} S_{n+1} \\ S_n \end{bmatrix} = \boldsymbol{M} \begin{bmatrix} S_n \\ S_{n-1} \end{bmatrix},$$

   where $\boldsymbol{M}$ is a matrix you have to determine using the recurrence relation. Then using the fact that $\begin{bmatrix} S_n \\ S_{n-1} \end{bmatrix}$ depends on $\begin{bmatrix} S_{n-1} \\ S_{n-2} \end{bmatrix}$, and so on, find a relation between $\begin{bmatrix} S_{n+1} \\ S_n \end{bmatrix}$ and $\begin{bmatrix} S_2 \\ S_1 \end{bmatrix}$, where $S_1 = S_2 = 1$. Use this relation to find an explicit formula for S_n.

3. Let $\boldsymbol{C} \in \text{Sym}$ be invertible, let $I_i \equiv I_i(\boldsymbol{C})$, and let (25)

   $$W(\boldsymbol{C}) = c_1(\bar{I}_1 - 3) + \frac{\kappa}{2}(J-1)^2, \quad J := \det \boldsymbol{F},$$

   where $\bar{I}_1 = (I_3)^{-1/3} I_1$, and $\kappa = \lambda + 2\mu/3$ is the bulk modulus. Using $\boldsymbol{S}(\boldsymbol{C}) = 2\partial W/\partial \boldsymbol{C}$, find an expression for $\boldsymbol{S}(\boldsymbol{C})$. Next, find $\boldsymbol{S}(\boldsymbol{I}) + D\boldsymbol{S}(\boldsymbol{I})[2\boldsymbol{E}]$ (first find $\boldsymbol{S}(\boldsymbol{I}) + D\boldsymbol{S}(\boldsymbol{I})[\boldsymbol{U}]$, and then put $\boldsymbol{U} = 2\boldsymbol{E}$), and using the fact that this should match with $\lambda(\operatorname{tr}\boldsymbol{E})\boldsymbol{I} + 2\mu\boldsymbol{E}$, find a relation between c_1 and μ.

4. Given that $\textbf{cof}\,\boldsymbol{F} : \boldsymbol{\nabla}_X \tilde{\boldsymbol{v}} = 0$, find the value of $\det \boldsymbol{F}$ as a function of time. (Hint: Try to express $\boldsymbol{\nabla}_X \tilde{\boldsymbol{v}}$ in terms of $\boldsymbol{F}$). (20)

Some relevant formulae

$$\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{w},$$

$$(\boldsymbol{a} \otimes \boldsymbol{b})(\boldsymbol{c} \otimes \boldsymbol{d}) = (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a} \otimes \boldsymbol{d},$$

$$\det(\boldsymbol{T} + \boldsymbol{U}) = \det \boldsymbol{T} + \textbf{cof}\,\boldsymbol{T} : \boldsymbol{U} + \textbf{cof}\,\boldsymbol{U} : \boldsymbol{T} + \det \boldsymbol{U},$$