Indian Institute of Science, Bangalore
ME 243: Midsemester Test

Date: 7/10/10.
Duration: 11.30 a.m.–1.00 p.m.
Maximum Marks: 100

1. Answer the following (*justify* all your steps):

   (a) If $(\boldsymbol{u}, \boldsymbol{T}_1\boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{T}_2\boldsymbol{v})$ for all $\boldsymbol{u}$, $\boldsymbol{v}$ which are linearly dependent, then is $\boldsymbol{T}_1 - \boldsymbol{T}_2$ equal to zero? If not, what is the most general form that $\boldsymbol{T}_1 - \boldsymbol{T}_2$ can have? (15)

   (b) If $(\boldsymbol{u}, \boldsymbol{T}_1\boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{T}_2\boldsymbol{v})$ for all $\boldsymbol{u}$, $\boldsymbol{v}$ which are mutually orthogonal, then is $\boldsymbol{T}_1 - \boldsymbol{T}_2$ equal to zero? If not, what is the most general form that $\boldsymbol{T}_1 - \boldsymbol{T}_2$ can have? (25)

2. Let $\{\boldsymbol{p}, \boldsymbol{q}, \boldsymbol{r}\}$ be an orthonormal basis. Find the factors $\boldsymbol{R}$ and $\boldsymbol{U}$ in the polar decomposition of $\boldsymbol{F} = \boldsymbol{p} \otimes \boldsymbol{p} + \gamma(\boldsymbol{q} \otimes \boldsymbol{r} - \boldsymbol{r} \otimes \boldsymbol{q})$, where γ is a nonzero number. Then determine the eigenvalues and the axis of $\boldsymbol{R}$. (20)

3. Let $\{\lambda_1, \lambda_2, \lambda_3\}$ be the distinct (not necessarily nonzero) eigenvalues of a tensor $\boldsymbol{T}$. (25)

   (a) Using $(\mathbf{cof}\,\boldsymbol{T})^T = I_2\boldsymbol{I} - (\operatorname{tr}\boldsymbol{T})\boldsymbol{T} + \boldsymbol{T}^2$, find the eigenvalues of $\mathbf{cof}\,\boldsymbol{T}$.

   (b) Using the above result, and the eigenvalues of $\boldsymbol{T} - \lambda_1\boldsymbol{I}$, find the eigenvalues of $\mathbf{cof}\,(\boldsymbol{T} - \lambda_1\boldsymbol{I})$.

   (c) Using the fact that λ_1 satisfies the characteristic equation of $\boldsymbol{T}$, i.e., $\det(\boldsymbol{T} - \lambda_1\boldsymbol{I}) = 0$, find $\partial\lambda_1/\partial\boldsymbol{T}$. The denonimator in your expression should be an explicit function of the eigenvalues. By a permutation of indices, find $\partial\lambda_2/\partial\boldsymbol{T}$ and $\partial\lambda_3/\partial\boldsymbol{T}$.

   (d) Let $\{\lambda_1, \lambda_2, \lambda_3\}$ denote the eigenvalues of $\boldsymbol{U} := \sqrt{\boldsymbol{C}} = \sqrt{\boldsymbol{F}^T\boldsymbol{F}}$. If

   $$W(\boldsymbol{C}) = \sum_{i=1}^{3} \frac{\mu_i}{\alpha_i}\left[\lambda_1^{\alpha_i} + \lambda_2^{\alpha_i} + \lambda_3^{\alpha_i} - 3\right] + k_1 \log(\det\boldsymbol{C}),$$

   where μ_i, α_i, $i = 1, 2, 3$, and k_1 are constants, find $\boldsymbol{S} = 2\partial W/\partial\boldsymbol{C}$.

4. We have defined the velocity vector $\tilde{\boldsymbol{v}}$ in the Lagrangian setting as $\tilde{\boldsymbol{v}}(\boldsymbol{X}, t) = (\partial\boldsymbol{\chi}/\partial t)_X$. Is the velocity in the Eulerian setting $\boldsymbol{v}(\boldsymbol{x}, t)$ equal to $\partial\boldsymbol{\chi}^{-1}/\partial t)_x$. If yes, provide a proof, if not, provide a counterexample. Similarly, is $(\boldsymbol{F}^{-1})_{ij} = \partial\chi_i^{-1}/\partial x_j$? If yes, provide a proof, if not, provide the correct expression with proof. (15)

Some relevant formulae

$$\det(\boldsymbol{T} + \boldsymbol{U}) = \det\boldsymbol{T} + \mathbf{cof}\,\boldsymbol{T} : \boldsymbol{U} + \mathbf{cof}\,\boldsymbol{U} : \boldsymbol{T} + \det\boldsymbol{U},$$

$$I_2(\boldsymbol{T}) = \frac{1}{2}\left[(\operatorname{tr}\boldsymbol{T})^2 - \operatorname{tr}(\boldsymbol{T}^2)\right].$$